ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

ANALYSIS AND DESIGN OF ROBUST DISTURBANCE OBSERVERS

Ph.D. THESIS

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Department of Control and Automation Engineering

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SEPTEMBER 2023

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İSTANBUL TEKNİK ÜNİVERSİTESİ ★ LİSANSÜSTÜ EĞİTİM ENSTİTÜSÜ

DAYANIKLI BOZUCU GÖZLEYİCİLERİNİN ANALİZ VE TASARIMI

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To Nihan,

FOREWORD

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ABBREVIATIONS

DOB	**:** Disturbance Observer
LMI	**:** Linear Matrix Inequality
LQG	**:** Linear Quadratic Gaussian
PI	**:** Proportional Integral
SBD	**:** Soh-Barger-Dabke
SVD	**:** Singular Value Decomposition

SYMBOLS

$\mathbf{A(q)}$	:	Uncertain matrix
$\mathbf{A_0}$	:	Nominal Hurwitz matrix
$\overline{\beta}$	:	Conjugate of a scalar β
$\mathbf{D_n(j\omega)}$	:	Nominal denominator polynomial
$\mathbf{H_n}$	:	Set of all nxn Hermitian matrices
$\mathbf{j}$	:	Imaginary unit
$\mathbf{M}^*$	:	Conjugate transpose of a vector or a matrix M
Re	:	Real Part of the expression
Im	:	Imaginary Part of the expression
$\mathbf{G(s)}$	:	Continuous transfer function of the plant
$\mathbb{C}$	:	Set of complex numbers
$\mathbb{R}$	:	Set of real numbers
$\mathbf{D^C}$	:	Complementary D-Region
$\mathbf{s}$	:	Complex argument for the Laplace transform
$\mathbf{u(t)}$	:	Control signal
$\mathbf{y(t)}$	:	System output
$\mathbf{q_i}$	:	Uncertain parameters
$\mathbf{P_n}$	:	Transfer Function of nominal plant
$\mathbf{P}$	:	Transfer Function of uncertain plant
$\mathbf{P_0(j\omega)}$	:	Center of the ellipse
θ	:	Hermitian matrix defining D_Θ D-region
$\mathbf{W}$	:	Weight matrix represents the lengths of major and minor axex of the ellipse
$\mathbf{T_{yr}}$	:	Transfer Function from reference signal (r) to system output (y)
$\mathbf{T_{yn}}$	:	Transfer Function from noise input (n) to system output (y)
$\mathbf{T_{yd}}$	:	Transfer Function from disturbance input (d) to system output (y)
$\lambda_\mathbf{i}$	:	ith Eigenvalue
ξ	:	ith closed-loop eigenvector
ω	:	Imaginary part of poles in continuous-time domain
$\otimes$	:	Kronecker product
$\oplus$	:	Kronecker sum
$\|\|\cdot\|\|$	:	Norm of $\cdot$

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SUMMARY

Robustness has been one of the most defining features of control systems since the classical control period. In the early days, the robustness of the control system was expressed using concepts like phase margin and gain margin, adapted from telecommunications engineering, and this terminology was faithfully used during the period when the significant achievements of modern control theory were demonstrated. However, by the end of the 70s, two separate developments marked the beginning of the golden age of robust control theory. The first of the developments that heralded this new era is Kharitonov's theorem, which established a new field of research for examining the stability of systems with parametric uncertainty. The other is John Doyle's demonstration that even in a single-input, single-output system, the LQG regulator does not have any guaranteed robustness margin, unlike the LQ regulator. While the first formed the basis of the research field known as the parametric approach, the other was one of the precursors of the H_∞ theory.

Since then, robust control has been seen as an independent sub-branch of control theory. Both approaches reached their peak with both theoretical and practical applications throughout the 1980s and 1990s. On the other hand, it has been shown that more robust closed-loop systems can be developed by changing the structure of the controller. One of the prominent methods is the approach known in the literature as the disturbance-observer (DOB). This approach, which enables the prediction and cancellation of disturbances and uncertainties that impact the system at its input, has been widely implemented, particularly in practical applications. On the other hand, the theoretical limits of the method, its analysis under uncertainty, and its design with newly developed robust control methods have lagged behind practical applications. Although theoretical studies have been carried out especially with the H_∞ approach since the 2000s, DOB design and analysis under parametric uncertainties have not attracted the attention of researchers sufficiently. The main purpose of this thesis is to develop new approaches for both the analysis and design of disturbance observers under parametric uncertainties.

In the analysis of systems with parametric uncertainty, how the uncertainties are modeled is the factor that directly affects the analysis method. In Kharitonov's paradigm, the parametric uncertainty bounding set is usually expressed as a box, which corresponds to the l_∞ representation of the parameter box. However, the l_2 analog of the same representation is also possible. In fact, this representation is more suitable for the situation where the mathematical model is obtained by linear or nonlinear regression methods under system identification approach. Based on this, in the first part of the thesis, the answer to the question of "How much uncertainty can be tolerated with the DOB structure?", has been sought.

Although approaches in the frequency domain produce effective results for DOB analysis, new challenges arise when the problem is expressed in the state space. Two approaches have come to the fore for examining parametric uncertainties in the state space. The first of these is to move the problem to the frequency domain where there are theorems and mathematical tools mature enough to examine parametric uncertainties. However, when this method is utilized, even the simplest interval system matrices show themselves as a affine-linear or more complex polynomial when expressed as a polynomial. Therefore, design in state space was seen as a "hard nut to crack" problem, in Yedevalli's words, and pushed control theorists to different research directions. The other method is to consider the problem directly in the state space. Although similar difficulties exist in this approach, when designing directly in the state space, the use of proven state space methods is also possible. Although new solutions are proposed, especially under the concept of quadratic stability, the nature of the problem condemns control theorists to use conservative approaches. In addition, a suitable Lyapunov function has not yet been proposed in the case where the design regions used to limit the parametric uncertainties are disjoint. The second contribution put forward within the scope of the thesis is the guardian-map approach, which offers less conservative disturbance observer design. Thanks to the method, robustness criteria can be assigned for each nominal eigenvalue separately and the disturbance observer is designed to meet this criterion. In this way, the inherent trade-off between robustness of the disturbance observer and the disturbance observer bandwidth is decided according to whether the closed-loop system satisfies the previously determined eigenvalue spread criterion.

Advantages of considering the problem in state space include the possibility to use LMI tools and the incorporation of useful methods such as eigenstructure assignment into the solution of the problem. Many control problems can be expressed in LMI form, and these LMIs can be formulated as appropriate convex optimization problems. The LMI framework is particularly useful for expressing parametric uncertainties and constraining eigenvalue spread. However, when the dominant methods in the literature are examined, the design regions defined by the LMI approach are not defined separately for each eigenvalue, but a combined LMI design region is defined for all eigenvalues. This situation complicates the eigenvalue assignment problem and does not allow defining different robustness criteria between the eigenvalues in the non-dominant region, which is less important for the design, and the dominant region eigenvalues, which determine the behavior of the system.

In addition, when the eigenstructure assignment methods are considered, the methods for minimizing the sensitivity of the system dominate the literature, instead of expressing the parametric uncertainties directly. Although robust eigenstructure assignment methods based on H_∞-based approaches have been proposed, eigenstructure assignment methods have not been sufficiently studied in direct parametric uncertainty system design. In the eigenstructure assignment methods, since the eigenvalues are assigned strictly at the beginning of the design, the vector space to which the eigenvectors can be assigned in the rest of the design is also limited. In order to overcome this, although methods such as regional assignment, partial eigenvalue assignment and loose eigenstructure assignment are suggested in the literature, suppressing the effect of parametric uncertainties has not been the primary design criterion in these approaches.

In order to fill these gaps in the literature, a new design method has been proposed, and in this approach, the robustness of the system to parametric uncertainties has been made the primary criterion of the design, and a novel disturbance observer design method has been proposed by using eigenstructure assignment and LMI approaches together for this purpose. The approach does not require any heuristic algorithms or global optimization methods, as well as allowing the solution of the robust root clustering problem for disjoint design regions. As a result, the method inevitably suffers from conservatism. However, the design reduces the problem of finding robust eigenvectors to finding the appropriate one among a finite number of eigenvectors.

As a conclusion, within the scope of this thesis, a method is proposed to examine the robustness of the disturbance observer under parametric uncertainties, and two new design methods are proposed to limit the eigenvalue spread in the state space within the disjoint design regions determined for each nominal eigenvalue. By using the obtained results, a disturbance observer in the state space is designed for systems with parametric uncertainty and the results are shared.

DAYANIKLI BOZUCU GÖZLEYİCİLERİNİN ANALİZ VE TASARIMI

ÖZET

Dayanıklılık klasik kontrol döneminden beri kontrol sistemlerinin en belirleyici özelliklerinden olmuştur. İlk dönemlerde kontrol sisteminin dayanıklılığı telekomünikasyon mühendisliğinden uyarlanan faz marjini ve kazanç marjini gibi terimler ile ifade edilmiştir ve bu terminoloji modern kontrol teorisinin en parlak başarılarının ortaya konduğu dönemde sadakatle kullanılmıştır. Ancak 70lerin sonuna gelindiğinde dayanıklı kontrol teorisinin altın çağının başlangıcı iki ayrı koldan ilan edilmiştir. Bu yeni dönemi müjdeleyen gelişmelerden ilki, tamamen yeni bir araştırma alanının yolunu açan ve parametrik belirsizlikli sistemlerin kararlılığının incelenmesine olanak sağlayan Kharitonov teoremidir. Diğeri ise, John Doyle'un, tek girişli, tek çıkışlı bir sistemde bile, LQ regülatörünün aksine, LQG regülatörünün herhangi bir garantili sağlamlık marjına sahip olmadığını göstermesidir. İlki parametrik yöntem olarak bilinen araştırma alanının temelini oluştururken diğeri H_∞ teorisinin öncüllerinden biri olmuştur.

Bu tarihten itibaren dayanıklı kontrol, kontrol teorisinin müstakil bir alt dalı olarak görülmüştür. Her iki yaklaşımda 80'li ve 90'lı yıllarda teorik ve pratik uygulamalarla zirveye ulaşmıştır. Ancak bir taraftan da kontrolörün yapısını değiştirerek daha dayanıklı kapalı çevrim sistemlerin geliştirilebileceği gösterilmiştir. Öne çıkan yöntemlerden biri literatürde bozucu gözleyicisi olarak bilinen yaklaşımdır. Sisteme etki eden bozucuların ve belirsizliklerin bir şekilde kestirilip sistem girişinde iptal edilmesine imkân tanıyan bu yaklaşım özellikle pratik uygulamalarda kendisine çok fazla yer bulmuştur. Buna mukabil yöntemin teorik sınırları, belirsizlik altında analizi ve yeni gelişen dayanıklı kontrol yöntemleri ile tasarımı gibi konular pratik uygulamaların gerisinde kalmıştır. Her ne kadar 2000'li yıllardan itibaren özellikle H_∞ yaklaşımı ile teorik çalışmalar ortaya konsa da parametrik belirsizlikler altında DOB tasarımı ve analizi konuları araştırmacıların ilgisini yeterince çekmemiştir. Bu tezin temel amacı bozucu gözleyicilerinin parametrik belirsizlikler altında hem analizi hem de tasarımı için yeni yaklaşımlar geliştirmektir.

Parametrik belirsizlikli sistemlerin analizinde belirsizliklerin nasıl modellendiği analiz metodunu doğrudan etkileyen faktördür. Kharitonov'un önünü açtığı yaklaşımda parametrik belirsizlik sınırlama kümesi genellikle kutu şeklinde ifade edilir bu da parametre kutusunun l_∞ temsiline denk gelir. Hâlbuki ki aynı temsilin l_2 analoğu da mümkündür. Hatta bu temsil, matematiksel modelin sistem tanıma altında lineer veya nonlineer regresyon yöntemleri ile elde edildiği duruma daha uygundur. Buradan yola çıkılarak tezin ilk bölümünde frekans tanım bölgesinde "DOB yapısı ile ne kadar belirsizlik tolere edilebilir?" sorusuna cevap aranmıştır.

Her ne kadar frekans tanım bölgesindeki yaklaşımlar DOB analizi için etkili sonuçlar üretse de problem durum uzayında ifade edildiğinde yeni zorluklar ortaya

çıkmaktadır. Durum uzayında parametrik belirsizliklerin incelenmesi için iki yaklaşım ön plana çıkmıştır. Bunlardan ilki problemi parametrik belirsizliklerin incelenmesi için yeterince güçlü teoremlerin ve matematiksel araçların olduğu frekans tanım bölgesine taşımaktır. Hâlbuki bu yöntem ile ilerlendiğinde en basit interval sistem matrisleri bile polinom olarak ifade edildiğinde kaymış lineer veya daha karmaşık yapıda bir polinom olarak kendini göstermektedir. Dolayısı ile durum uzayında tasarım Yedevalli'nin deyimi ile "çetin ceviz" bir problem olarak görülmüş ve kontrol teorisyenlerini farklı arayışlara itmiştir. Diğer yöntem ise problemi doğrudan durum uzayında ele almaktır. Benzer zorluklar bu yaklaşımda da mevcut olmasına rağmen doğrudan durum uzayında tasarım yapıldığında başarısı ispatlanmış durum uzayı yöntemlerinin kullanılmasının da önü açılmaktadır. Özellikle quadratic kararlılık konsepti altında yeni çözümler önerilse de problemin doğası kontrol teorisyenlerini konservatif yaklaşımlara mahkum etmektedir. Ayrıca parametrik belirsizliklerin sınırlandırılması için kullanılan tasarım bölgelerinin ayrık olduğu durumda uygun bir Lyapunov fonksiyonu henüz önerilememiştir. Tez kapsamında öne sürülen ikinci yenilik daha az tutuculuk öneren guardian-map yaklaşımı ile bozucu gözleyicisi tasarımıdır. Yöntem sayesinde her bir nominal özdeğer için ayrı ayrı dayanıklılık kriteri atanabilmekte ve bozucu gözleyicisi bu kriteri sağlamak için tasarlanmaktadır. Bu sayede bozucu gözleyicisinin yapısı gereği var olan dayanıklılık ve DOB bant genişliği arasındaki ödünleşmede, kapalı çevrim sistemin daha önce belirlenen özdeğer saçınım kriterini sağlayıp sağlamamasına göre karar verilmektedir.

Problemi durum uzayında ele almanın avantajları arasında LMI araçlarının kullanımının mümkün olması ve özdeğer-özvektör ataması gibi kullanışlı yöntemlerin problemin çözümüne dahil edilebilmesi yer almaktadır. Pek çok kontrol problemi LMI formunda ifade edilebilir ve bu LMI'lar uygun dışbükey optimizasyon problemleri olarak formüle edilebilir. Özellikle parametrik belirsizlikleri ifade etmek ve özdeğer saçınımını sınırlamak için LMI çerçevesi kullanışlıdır. Ancak literatürde baskın olarak kullanılan yöntemler incelendiğinde LMI yaklaşımı ile tanımlanan tasarım bölgeleri her bir özdeğer için ayrı ayrı tanımlanmamakta, tüm özdeğerler için birleşik bir LMI tasarım bölgesi tanımlanmaktadır. Bu durum özdeğer atama problemini zorlaştırmakta, tasarım için daha az öneme sahip baskın olmayan bölgedeki özdeğerler ile sistemin davranışını belirleyen baskın bölge özdeğerleri arasında farklı dayanıklılık kriterleri tanımlanmasına imkan vermemektedir.

Bunun yanında özdeğer-özvektör atama yöntemleri ele alındığında yaklaşım olarak parametrik belirsizliklerin doğrudan ifade edilmesi yerine sistemin duyarlılığını minimize etmeye yönelik yöntemler literatürü domine etmektedir. H_∞ tabanlı yaklaşımların temel alındığı dayanıklı özdeğer-özvektör atama yöntemleri önerilmiş olsa da doğrudan parametrik belirsizlikli sistem tasarımı konusunda özdeğer-özvektör atama yöntemleri yeteri kadar incelenmemiştir. Özdeğer-özvektör atama yöntemlerinde, yaklaşım olarak özdeğerler tasarımın başında kesin olarak atandığı için tasarımın geri kalanında özvektörlerin atanabileceği vektör uzayı da sınırlanmakta, dolayısı ile D-karalılık söz konusu olduğunda durum uzayı yaklaşımın tasarım serbestliği özdeğer-özvektör atama yöntemlerinde dayanıklılık lehine yeterince kullanılamamaktadır. Bunu aşmak adına literatürde bölgesel atama, kısmı özdeğer atama ve gevşek özdeğer-özvektör atama gibi yöntemler önerilse de bu yaklaşımlarda da parametrik belirsizliklerin etkisini bastırmak öncelikli tasarım kriteri olmamıştır.

Literatürde yer alan bu boşlukları doldurmak adına yeni bir tasarım yöntemi önerilmiş, bu yaklaşımda hem sistemin parametrik belirsizliklere karşı dayanıklılığı doğrudan tasarımın öncelikli kriteri haline getirilmiş hem de özdeğer-özvektör atama ve LMI yaklaşımları bu amaç için birlikte kullanılarak özgün bir bozucu gözleyicisi tasarım yöntemi ortaya konmuştur. Yaklaşım ayrık tasarım bölgeleri için dayanıklı kutup kümeleme (robust root clustering) probleminin çözümüne olanak sağlamanın yanı sıra herhangi bir sezgisel algoritma veya gobal optimizasyon yöntemine ihtiyaç duymamaktadır. Bunun bir sonucu olarak yöntem tutuculuktan kaçınılmaz olarak mustariptir. Ancak tasarım dayanıklı özvektörlerin bulunması problemini sonlu sayıda özvektör arasından uygun olanının bulunmasına indirgemiştir.

Özetle, bu tez kapsamında bozucu gözleyicisini parametrik belirsizlikler altında dayanıklılığının incelenmesi için bir yöntem önerilmiş, durum uzayında özdeğer saçınımını her bir nominal özdeğere özgü belirlenen ayrık tasarım bölgeleri içerisinde sınırlamak için iki yeni tasarım yöntemi önerilmiştir. Elde edilen sonuçlar kullanılarak parametrik belirsizlikli sistemler için durum uzayında bozucu gözleyicisi tasarlanmış ve sonuçlar paylaşılmıştır.

1. INTRODUCTION

1.1 Motivation

Robustness is one of the fundamental concepts that has occupied the minds of feedback control theorists since the classical control period. Since uncertainties and disturbances affect the system performance dramatically, the concept of robustness should be considered as one of the primary control system design criteria especially in applications that require aggressive tracking performance such as inertially stabilized platforms and motion control systems operating in harsh environments. In order to tackle these constraints, disturbance observer-based control systems are taking more attention. However, as it is indicated in [1], robustness analysis on DOB based control system still lacks detailed results, especially in parametric theory. Here, an answer to the question of " how much uncertainty can be dealt with the DOB structure?", in a systematic manner using a parametric approach is sought.

A suitable analysis framework is essential for putting forward the weaknesses and potentials of the disturbance observer based control systems. Although mainstream H_∞-based robust control methods were dominant in the literature, their complexity and conservative nature brought forward an alternative line of research called the parametric approach. The majority of the robustness analyses in DOB based control systems have stayed in the non-parametric area. Therefore, the first part of the study deals with the robustness analysis of DOB based control system using a spherical polynomial method. This approach enables the analysis of different cases where DOB based control systems lose their superiorities. By utilizing the value set concept, a graphical examination of the robustness analysis is validated. The theoretical results have been discussed on a motion control system model.

On the other hand, designing a robust disturbance observer-based control system has gained the attention of researchers since the early 2000s. However, the main driving force of those research directions is not to tackle the effect of parametric uncertainties. Although several attempts have been carried out based on the Kharitonov methods, theoretical studies still lag behind practical developments. One of the main reasons for these difficulties is to find a suitable framework for designing disturbance observer-based control systems under parametric uncertainties. Therefore, the second part of the thesis focuses on different design approaches that prioritize reducing the effects of parametric uncertainties as the main design objective.

1.2 Literature Review

1.2.1 Robust control

In the 1970s, it was discussed aloud that LQG control has robustness problems against changes in system parameters and unmodeled dynamics [2]. Without a doubt, the paper written by Doyle [3] , which is one of the most influential publications on the subject, has opened an important gateway for robust control system research. In parallel with the discussion of how to modify LQG, new optimization problems based on the H_∞ norm, have been proposed [4]. It is known that the H_∞ norm is closely related with the largest singular value of the frequency response of the system and hence the worst-case amplification of the error [5], [6].

Almost at the same period as Doyle's paper, [3], an important discovery was made by Kharitonov, who showed a way of determining the stability of an interval polynomial family by examining only four vertex polynomials [7]. This progress led to another branch of robust control theory called the parametric approach. Following years, many researchers contributed to this elegant theory. In 1989, Mansour showed that fewer than four polynomials are enough to determine the stability if the order of the polynomial is less than 6 [8], Barlett, Hollot and Huang showed that edge polynomials must be checked to determine the stability of affine linear polynomials [9]. A generalization of Kharitonov's findings is examined in [10], [11]. Although Kharitonov-based methods [7] open up new horizons in designing systems with parametric uncertainty in the frequency domain, the available tools do not show the same success when it comes to matrix families. As Yedevalli stated in [12], checking robust stability in matrix families is a tougher problem than checking in polynomial families. The study in [13] may give a profound insight into the topic.

Since DOB based control methods are seldomly examined from the parametric point of view one of the aims of this study is to contribute to the DOB analysis literature by utilizing the parametric approach. Another aim of the study is to propose a new eigenstructure assignment method for restricting closed-loop eigenvalue spread in disjoint regions for matrices with parametric uncertainty, as well as to bring a different approach to the design of the disturbance observer in the state space.

1.2.2 Disturbance observer

It is not surprising to those who are interested in the history of control that the intellectual foundations of many of the control methods and ideas we use today have been introduced in the years of modern control theory. The basis of the disturbance observer idea dates back to the 1960s, justifying this determination [14]. Johnson's proposed structure [15] for estimating fixed disturbances acting on the system within the framework of optimal control can be considered the basis of the disturbance observer idea used today. His other publication [16], which proposes to predict the external disturbances that affect the system besides the states of the system, includes the idea known as the extended state observer today.

The disturbance observer structure, as used today, was first proposed to reduce the sensitivity to a parameter change, nonlinear effects, and other disturbances in servo motor control [17] and was used in the motion control framework in [18].

In conventional control applications, the disturbance is used to express the uncontrollable quantities that affect the system from the outside. With this definition, disturbance rejection performance is considered one of the basic design criteria for many methods, from loop shaping to H_∞ [19]. However, some approaches that treat parametric uncertainties and unmodeled dynamics of the system as disturbance [20]. The reader may found an extensive discussion on this topic in [19]. To the best of the authors' knowledge, the most general and inclusive approach to the nature of the disturbance is given by the concept of active disturbance rejection proposed by Han. According to this approach, there is no need for a detailed mathematical model of the system [19]. If the term disturbance is used in the most general sense, the control problem becomes a disturbance suppression problem [21], [22].

Many researchers have described similar ideas at different times with different names, and there has been an overwhelming accumulation of DOB-based methods in the literature. The reader should refer to [23] to evaluate the observer-based approaches such as “unknown input observer”, “equivalent input disturbance”, “extended state observer”, “generalized PI observer” [24], “disturbance and uncertainty estimation”, “active disturbance rejection”. In [25], the advantages and disadvantages of many observer based methods used in practical applications are discussed.

The success of DOB-based methods in practical applications has paved the way for theoretical analysis. Arguably, it can be said that theoretical analysis follows practical application in the development of the disturbance observer-based approach. DOB is used in many industrial systems such as automatic steering of a vehicle [26], CNC machines [27], mechanical positioning systems [28], piezoelectric actuators [29], and attitude control of a missile [30]. Although DOB has found enough application areas in practice, the literature on theoretical analysis is still not satisfactory [23], [31]. Therefore, different approaches are still brought about for issues such as robust stability, the effect of right half plane zeros and poles, and time-delayed systems. The performance and stability of DOB are studied in [32] using singular perturbation theory, with similar results proven in the frequency domain [33]. The robust stability conditions for minimum phase systems under parametric uncertainties are given in [34]. In [35], it has also been shown that the phase and gain margins for minimum phase systems can be increased arbitrarily using a DOB based approach. DOB is also recommended for non-minimum phase systems [36–38]. In [39], a reduced-order DOB based method is proposed to obtain a low order observer. An approach on the case when the difference between denominator and numerator of the uncertain system is not known exactly is proposed in [40, 41]. Existingly, the criteria for robust stability and robust performance are reduced to the bandwidth of the Q filter [42, 43]. H_∞ based and conservative methods predominate in the literature [43–45]. Especially analyzes under parametric uncertainty are limited [34], [46] .

1.2.3 Eigenstructure assignment for robust control

Studies on the eigenstructure assignment problem usually considered to begin with the work of Porter and Crossley [47]. Especially after the studies [48], [49] which showed the connection of controllability with the eigenspectrum, the research direction shifted to questions about the relationship between the number of input/output and degrees of freedom of the system. [50], [51] and [52] have shown that $(m+l-1)$ eigenvalues can be arbitrarily assigned for a system with l output and m input. [53] provides a more extensive development of these results.

In the late 70s, with the works of [54], [55] assignment of the eigenstructure (eigenvalues and eigenvectors) become one of the main canonical design methods which is utilized in many time-domain problems today. On the other hand, studies on the robustness performance of the method [56], [57] progressed in parallel. When it comes to robustness, the most prominent method in the eigenstructure assignment literature is the method known as the orthogonal eigenvector method [58]. At the end of this line of research, it is pointed out that condition numbers which are obtained by the right and left eigenvectors are closely related to the robustness of the system [59]. Especially with the maturation of the H_∞ theory, another robust eigenstructure assignment approach has been proposed [60]. Considering the proposed methods, the eigenstructure assignment problem requires a two-stage design. In the first stage, the eigenvalue (or eigenvector) is assigned from the beginning. Then, the relevant eigenvector (or eigenvalue) assignment is made.

However, in the case where the eigenvalues are strictly assigned beforehand, the relevant eigenvectors must lie in the subspace spanned by the columns of $(\lambda_i I - A)^{-1}B$, so the eigenvalues that are assigned strictly at the beginning of the procedure restrict the design.

Nevertheless, the exact eigenvalue assignment is not always necessary. Therefore, methods that do not assign the eigenvalues exactly are also suggested [61]. This brings the eigenstructure assignment literature to the concept of D-stability. At this point, to obtain more design freedom, strict partial eigenvalue assignment has been proposed by [62] and a regional assignment method is proposed by [63].

In particular, the D-stability problem, in which the eigenvalues are aimed to remain within a certain region, has gained significant importance in the LMI framework [64]. This method, in which all eigenvalues are restricted within a single convex region, has found a place for itself, especially in multi-purpose control applications [65]. However, these methods cluster all closed-loop eigenvalues in a connected region and do not consider the eigenstructure assignment. When the assignment regions are not common and disjoint, the assignment conditions cannot be reduced to a well-known Lyapunov inequality. This is the main difficulty of working with the disjoint regions.

In [66], authors consider the eigenvalue location problem as a mere quadratic optimization problem. Then, the quadratic problem is formulated as an LMI problem with a non-convex rank constraint. This result paved the way for representing disjoint design regions in the LMI framework. In [67], it is shown that rank condition is not necessary if the design region is convex. In [68], an elegant method is proposed for eigenstructure assignment for disjoint regions. However, the method utilizes a non-convex optimization method to handle robustness concerns. Also, the method mentioned for the nominal system is not suitable for systems with parametric uncertainties.

This study aims to cluster uncertain eigenvalues into a disjoint region by using an eigenstructure assignment. In the eigenstructure assignment, eigenvalues are assigned strictly to desired points. However, instead of assigning eigenvalues to specific points, designers may utilize design regions so that there is much more design freedom left for further control objectives. Instead of using non-convex heuristic algorithms to tackle the problem, we introduce a dual-stage design consisting of simple LMIs, with a reasonable conservatism.

1.2.4 Disturbance observer design

Similar to the eigenstructure assignment literature, the roots of disturbance observer techniques date back to the late 60s [69]. The idea of predicting external and internal disturbances led researchers to the extended state observer [16]. In [17], Ohishi introduces the structure of the DOB to tackle the problem of attenuating the effects of parameter changes, disturbances, and nonlinear effects.

A similar philosophy is expressed by the researchers such as [19] in the context of active disturbance rejection, in which uncertainties and unmodelled dynamics are treated as a disturbance. If the most inclusive manner of the term disturbance is used, the controller design problem becomes a disturbance attenuation problem, as indicated in [21].

Since the idea of disturbance rejection found extensive use for itself in different contexts of the control theory, literature is overwhelmed by the observer-based approaches such as, “unknown input observer”, “extended state observer”, “disturbance and uncertainty estimation”, and “active disturbance rejection” despite being different extensions of the same idea [25]. Although DOB-based methods find an extensive application area in the industry, the design and analysis methods in the theoretical domain still lack profoundness as it is stated in [70] and [31]. The robustness properties of the DOB have been investigated by several researchers, mainly since the 2000s. In [32], the robust stability of DOB in the state-space application is investigated by singular perturbation theory, besides similar outcomes obtained in the frequency domain [33]. Research in the direction of DOB design under parametric uncertainty in the frequency domain is not yet at a sufficient level. Studies in [34,42,71] are excellent exceptions.

The author of this study has also contributed to the analysis of DOB-based control systems via spherical polynomials [72]. However, to the best of the author's knowledge, there is no detailed study in the context of eigenstructure assignment for the DOB-based control system under parametric uncertainties. One of the aims of this study is to show that, extended loose eigenstructure structure assignment put forward a viable solution to the problem in the time domain.

1.3 Goal and Unique Aspect of the Thesis

The first part of the thesis proposes a new analysis method for disturbance observer-based control systems under parametric uncertainties. The study makes several key contributions including the derivation of the analytical relationship between the bandwidth and robustness of the disturbance observer using a spherical polynomial representation. Additionally, the study introduces the spherical value set approach for uncertain polynomials, which is applied for the first time to disturbance observer-based control systems. The study also presents the first systematic statement of the robustness margin for a given DOB-based system in the context of spherical polynomial families. Furthermore, the study examines the non-minimum phase case and discusses bandwidth constraints, as well as the effects of low-order DOB filter design. Overall, the study provides valuable insights into the design and analysis of disturbance observer-based control systems under parametric uncertainties.

The second part of the study concentrates on designing disturbance observer-based control systems under parametric uncertainties in the state space. To this end, a guardian-map-based disturbance observer design method is introduced first. The proposed method enables the clustering of perturbed eigenvalues into predefined disjoint regions.

The final part of the study proposes another method for designing disturbance observer-based control systems under parametric uncertainties, using eigenstructure assignment in the LMI framework. The proposed method has several novel aspects. Firstly, it allows for the handling of robust root clustering problems for disjoint D-regions in the context of eigenstructure assignment. Secondly, the method extends the capability of loose eigenstructure assignment procedures to uncertain matrix families. The method does not require any global optimization methods or heuristic algorithms, with only simple LMIs being required to find robust controllers. Thirdly, finding robust eigenvectors is reduced to selecting a finite number of alternatives. Finally, the method proposes a DOB design method for parametric uncertain systems in the context of eigenstructure assignment. Overall, these novel aspects of the method aim to provide an effective and flexible approach to designing disturbance observer-based control systems under parametric uncertainties.

1.4 Structure of the Thesis

Chapter 1 of the thesis provides an introduction to the research topic, including the motivation and goals of the study. The unique aspects of the thesis are also discussed, along with a detailed literature survey that provides a background for the research. This chapter serves as an overview of the thesis and sets the stage for the subsequent chapters. It provides a context for the research and lays out the research questions and objectives. Additionally, the chapter highlights the significance of the research and the potential contributions to the field. The chapter concludes by providing a brief outline of the structure of the thesis and the organization of the subsequent chapters.

Chapter 2, titled "Analysis of DOB-based Systems with Parameter Uncertainty", starts with introducing the DOB structure in frequency domain. The chapter provides preliminary analysis and required mathematical tools in Section 2.1. The results on the interval polynomial case are given in Section 2.2 and on the affine-linear polynomial case are given in Section 2.3.

Chapter 3, titled "Guardian-Map for Robust D-Stability", gives the state space representation of the DOB-based control system and introduces a new guardian map for the robust D-stabilization. The state feedback design method based on the proposed method is given in Section 3.4 and DOB design in this context is examined in Section 3.5.

Finally, Chapter 4, titled "Disturbance Observer Design by Extended Loose Eigenstructure Assignment for the Disjoint D-Region Stability", gives the novel design method for the DOB-based control systems under parametric uncertainty by utilizing eigenstructure assignment in LMI framework. Theoretical background is given in Section 4.2 and problem formulation is given in Section 4.3. Main results are shared in Section 4.4.

Thesis study is summarized in Chapter 5.

2. ANALYSIS OF DOB BASED SYSTEM WITH PARAMETER UNCERTAINTY [1]

2.1 Preliminaries

The basic DOB structure is given in Figure 2.1. In the figure, the uncertain plant is P, nominal plant model is P_n and the feedback controller is C. Finally, Q is a transfer function related with the disturbance observer, which is usually designed as a low pass filter. The relation between the inputs and the output of the system is given in (2.1).

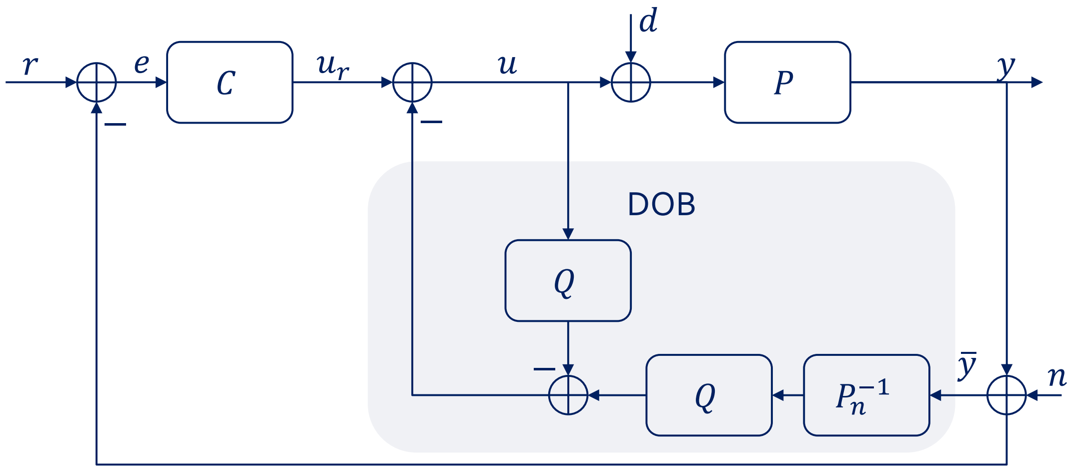

Figure 2.1 : Disturbance observer structure.

$$y = \frac{PP_n}{P_n + (P - P_n)Q} u_r + \frac{PP_n(1-Q)}{P_n + (P - P_n)Q} d - \frac{PQ}{P_n + (P - P_n)Q)} n \tag{2.1}$$

In (2.1), it is obvious that the transfer function from u_r to y is P_n if $P = P_n$. Therefore, when there is no uncertainty, the inner loop behaves like a nominal model for the input u_r. Also note that when $Q \approx 1$, the output of the system is almost decoupled from the input d. In the presence of the feedback controller C, the closed loop transfer functions are given below;

$$T_{yr} = \frac{P_n PC}{P_n(1 + PC) + Q(P - P_n)} \tag{2.2}$$

[1]This chapter is based on the paper "İsa Eray Akyol & Mehmet Turan Söylemez (2023) Analysis of disturbance observer-based control systems via spherical polynomials, International Journal of Control, 96:2, 435-448, DOI: 10.1080/00207179.2021.2000030"

$$T_{yn} = \frac{P(Q+P_nC)}{P_n(1+PC)+Q(P-P_n)} \tag{2.3}$$

$$T_{yd} = \frac{P_nP(1-Q)}{P_n(1+PC)+Q(P-P_n)} \tag{2.4}$$

Assuming that the amplitude of the transfer function Q is approximately 1 in a wide frequency band, the following relations are valid;

$$Q \approx 1 \implies T_{yr} \approx \frac{P_nC}{1+P_nC} \text{ and } T_{yd} = 0 \tag{2.5}$$

Considering the scheme in Figure 2.1, transfer functions between inputs $[r,d,n]^T$ and outputs $[e,u,\bar{y}]^T$ are given below;

$$\frac{1}{\sigma(s)}\begin{bmatrix} Q(P-P_n)+P_n & (Q-1)PP_n & (Q-1)P_n \\ CP_n & (1-Q)P_n & -Q-CP_n \\ CPP_n & (1-Q)PP_n & (1-Q)P_n \end{bmatrix} \tag{2.6}$$

where;

$$\sigma(s) = (1+PC)P_n+Q(P-P_n) \tag{2.7}$$

The matrix in (2.6) shows that, depending on the selection of Q, the nominal performance can be recovered and the effects of the disturbance d on the output can be cancelled provided that all nine transfer functions are stable.

2.1.1 Machinary

In the context of the parametric approaches, polynomial families are categorized by how uncertain parameters are affecting the coefficients of the polynomials. The reader may refer to [8] and [73] for further details.

Even if the plant polynomial has a simple structure such as interval type, the inner loop of the disturbance observer-based system may lead to a more complex polynomial family. This situation gets more complicated when the plant polynomial has already a complex structure. At this point, a certain conservatism may be accepted during the plant modeling in order to obtain a more refined solution. Therefore, interval polynomials with ellipsoidal uncertainty are considered mainly in this paper. The rest of this section covers uncertainty representation and the value set concept for the ellipsoidal polynomials.

2.1.1.1 Spherical versus box representation

It is natural to think that the uncertain parameters in real life (inertia, friction coefficient, etc.) are independent from each other.

Therefore, in the main streamline of the parametric approach, the uncertainty bounding set, Q, is assumed to be a box. In the literature, this assumption leads to l_∞ representation of the parameter box. However, it is also possible to use l_2 analogue of this representation. The comparison of sphere and box bonding set is given in Figure 2.2. In this paradigm, the uncertainty bounding set would be a hypersphere. Considering l_2 analog does not necessarily mean that the inertia and friction coefficient are related somehow. However, this becomes a meaningful approach specially when it is considered that the corners of the parameter box, in other words, the maximum and minimum values of the uncertain parameters may not be known exactly in real life. Ellipsoidal uncertainty representation particularly fits where the mathematical models

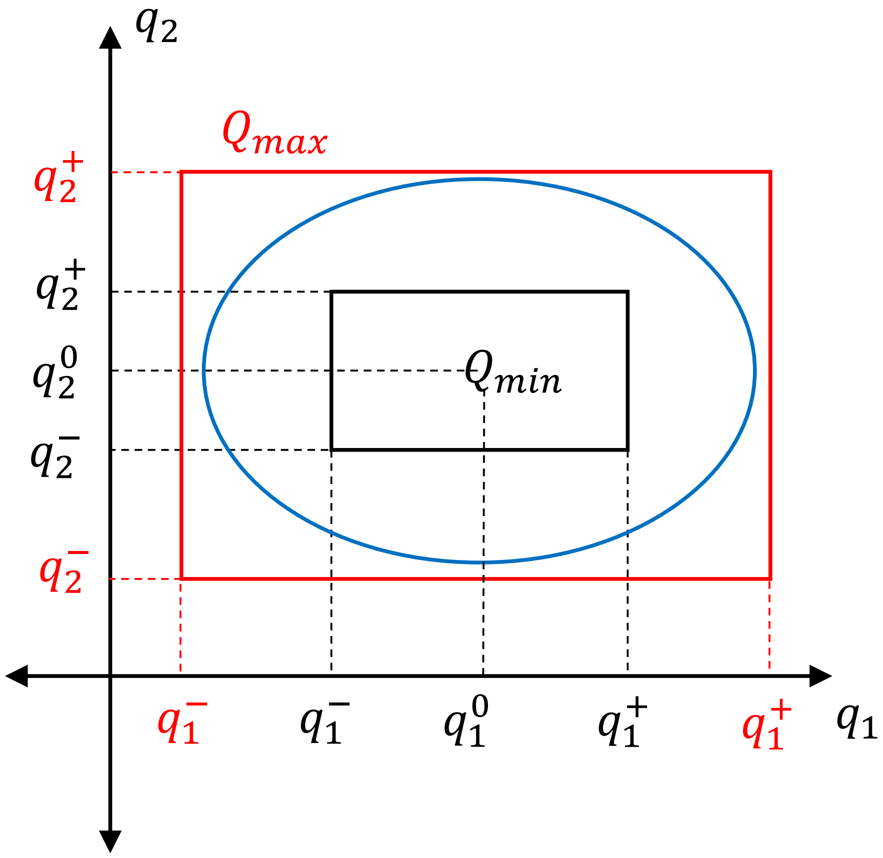

Figure 2.2 : Sphere vs. box bounding sets.

are obtained by linear or nonlinear regression techniques [74]. A connection of the system identification and robust controller design via ellipsoidal parametric uncertainty representation is given in [75].

SBD theorem [76], which is given as Theorem 2.1, is particularly useful for analyzing the systems with ellipsoidal uncertainty. The representation of Theorem 2.1 is directly obtained from [77]. Although it can only be used where the uncertain polynomial is interval type, the main idea behind the theorem can be extended to more complex representations.

Theorem 2.1 (SBD Theorem) [77]: *Consider the spherical family of polynomials P with invariant degree $n \geq 1$ described by*

$$p(s,q) = \sum_{i=0}^{n} q_i s^i + p_0(s) \tag{2.8}$$

with polynomial

$$p_0(s) = \sum_{i=0}^{n} a_i s^i \tag{2.9}$$

and uncertainty bounding set $\|q\|_2 \leq r$. For $\omega > 0$, let

$$G_{SBD}(\omega) = \frac{[Re\,(p_0(j\omega))]^2}{\sum_{i_{even}} \omega^{2i}} + \frac{[Im\,(p_0(j\omega))]^2}{\sum_{i_{odd}} \omega^{2i}} \tag{2.10}$$

Then P is robustly stable if, and only if, the following conditions are satisfied;

1) $p_0(s)$ is stable,

2) $|a_0| > r$

3) $G_{SBD}(\omega) > r^2, \forall \omega > 0$.

Theorem 2.1 will be the basis theorem for further theoritical development.

2.1.1.2 Value set for spherical polynomials

The value set concept with the zero exclusion principle brings important machinery to analyze the uncertain systems. It is already pointed out in [77] that the value set for a spherical polynomial at a given frequency $\omega > 0$ is an ellipse on the complex plane which can be expressed in the following form;

$$P_0^T(j\omega) W P_0(j\omega) \leq r^2 \tag{2.11}$$

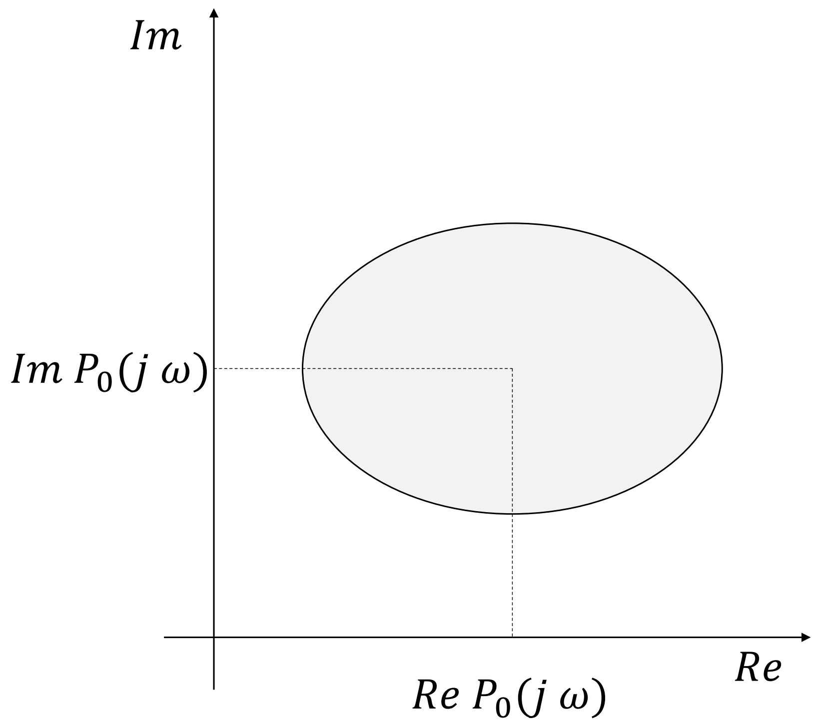

Figure 2.3 : An ellipsoidal value set for a single frequency $\omega > 0$.

where W is the weight matrix of the ellipse and the vector $P_0(j\omega)$ determines the center of the ellipse as depicted in Figure 2.3. An important observation is that the weight matrix is diagonal when the characteristic polynomial is of interval type. In the affine linear polynomial case, the off-diagonal terms of the weight matrix are nonzero. The reader may refer to [77] and [78] for further derivation of the value sets for ellipsoidal uncertainty.

2.2 Interval Polynomials

In this section, the aim is to put an explicit relation between robustness and the disturbance observer bandwidth using a parametric approach. Note that for a special case in which second-order transfer function is considered, such a relation has been obtained in [34]. However, the general case is considered here.

To derive the relation let us make the following definitions;

$$P(s) = \frac{N_p(s)}{D_p(s)} \tag{2.12}$$

$$P_n(s) = \frac{N_n(s)}{D_n(s)} \tag{2.13}$$

$$Q(s) = \frac{N_q(s)}{D_q(s)} \tag{2.14}$$

Now, by considering the DOB based structure in Figure 2.1, the inner loop transfer function can be obtained as follows;

$$T_{u_r y}(s) = \frac{PP_n}{Q(P - P_n) + P_n} \tag{2.15}$$

By substituting the definitions into the $T_{u_r y}(s)$, the following transfer function is obtained;

$$T_{u_r y}(s) = \frac{D_q N_n N_p}{D_p D_q N_n - D_p N_n N_q + D_n N_p N_q} \tag{2.16}$$

At this point, an assumption has to be made in order to use SBD theorem since it is only applicable to interval polynomial families. If the plant uncertainty is present only in the denominator and the relative degree of the plant is 1 then the characteristic polynomial is an interval type. However, if uncertain parameters are at both numerator and denominator and/or the relative degree of the plant is greater than 1 then the characteristic polynomial type is affine linear [34]. In this section we focus on the interval characteristic polynomials. Two possible cases exist.

2.2.1 Case-1: Same orders of $P(s)$ and $P_n(s)$

The inner loop of DOB structure is given in Figure 2.4. The following assumptions are

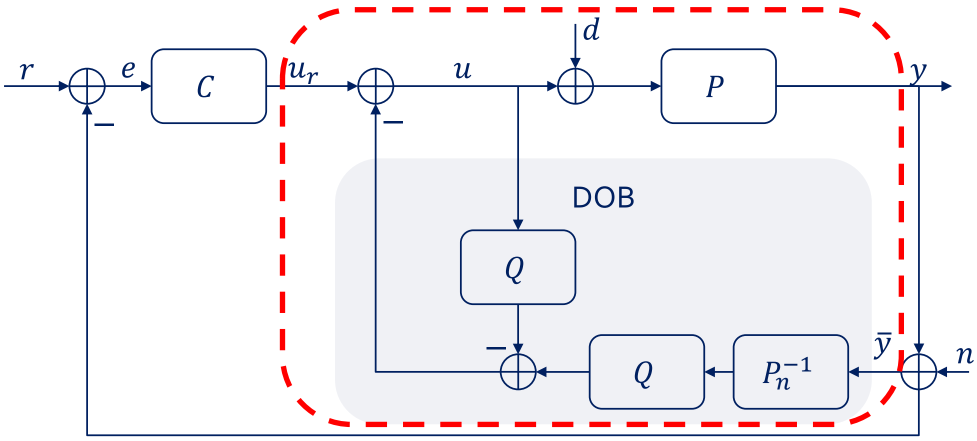

Figure 2.4 : Inner loop of DOB structure.

given in order to state Lemma 2.1.

Assumption 1: Only the denominator of P is uncertain and relative degree of P is 1.

Assumption 2: $P_n(s)$ is stable.

Lemma 2.1: *Consider the inner loop of DOB structure given in Figure 2.4 and the inner loop transfer function given in (2.16). Assume that, Under Assumption 1 and 2, as the bandwidth of the disturbance observer, g_0, increases, the robust stability margin of T_{u_ry} increases if the g_0 is higher than a certain value called its minimum value.*

Proof: Due to Assumption 1;

$$N_n = N_p \tag{2.17}$$

Hence, the characteristic polynomial of $T_{u_ry}(s)$ is reduced to;

$$P_{char}(s) = D_p(D_q - N_q) + D_n N_q \tag{2.18}$$

Furthermore, since the relative degree of the plant is 1, Q filter shall be selected as a first order low pass filter. Without losing generality, monic polynomials are used in the following.

$$Q(s) = \frac{g_0}{s+g_0} \tag{2.19}$$

$$\frac{N_n(s)}{D_n(s)} = \frac{b_k s^k + b_{k-1}s^{k-1} + \ldots + b_0}{s^{k+1} + a_k s^k + \ldots + a_0} \tag{2.20}$$

$$\frac{N_p(s)}{D_p(s)} = \frac{b_k s^k + b_{k-1}s^{k-1} + \cdots + b_0}{s^{k+1} + (a_k + \Delta a_k)s^k + \cdots + (a_0 + \Delta a_0)} \tag{2.21}$$

Here, $D_p(s)$ can be factorized as follows;

$$D_p(s) = D_{p0}(s) + D_{p\Delta}(s) \tag{2.22}$$

where;

$$D_{p0}(s) = s^{k+1} + a_k s^k + \ldots + a_0 = D_n(s) \tag{2.23}$$

$$D_{p\Delta}(s) = \Delta a_k s^k + \ldots + \Delta a_0 \tag{2.24}$$

Finally, the closed-loop system characteristic polynomial is obtained as follows;

$$P_{char}(s) = [D_{p0}(s) + D_{p\Delta}(s)]s + g_0[D_n(s)] \tag{2.25}$$

The closed-loop characteristic polynomial for the nominal system ($P_{char,0}(s)$) is then found as

$$\begin{aligned} P_{char,0}(s) &= D_{p0}(s)s + g_0 D_{p0}(s) && (2.26)\\ &= (s^{k+1} + a_k s^k + \ldots + a_0)(s+g_0) && (2.27) \end{aligned}$$

Now, due to Theorem 2.1; $D_{p0}(s)(g_0+s)$ must be stable. Assumption 2 states that $D_{p0}(s)$ is stable. Therefore, the first condition of Theorem 2.1 satisfied for $g>0$. Now, the second condition of Theorem 2.1 requires that;

$$|a_0 g_0| > r \tag{2.28}$$

Obviously, as the bandwidth of the disturbance observer is increased, this condition is satisfied more easily. Finally, the last condition of Theorem 2.1 requires that,

$$G_{SBD}(\omega) = \frac{[Re(D_{p0}(j\omega)(g_0+j\omega))]^2}{\sum_{i_{even}} \omega^{2i}} + \frac{[Im(D_{p0}(j\omega)(g_0+j\omega))]^2}{\sum_{i_{odd}} \omega^{2i}} > r^2 \tag{2.29}$$

Let;

$$D_{p0}(j\omega) = Re(D_{p0}(j\omega)) + jIm(D_{p0}(j\omega)) \tag{2.30}$$

Then, the numerator of the first part of $G_{SBD}(\omega)$ is;

$$\begin{aligned} \left[Re(D_{p0}(j\omega)(g_0+j\omega))\right]^2 &= (Re(D_{p0}))^2 g_0^2 \\ -2(Im(D_{p0})Re(D_{p0}))\omega g_0 &+ (Im(D_{p0}))^2\omega^2 \end{aligned} \tag{2.31}$$

Note that this is a quadratic function of g_0 and $(ReD_{p0})^2 > 0$. Obviously for positive g_0, the value of this function is increased as g_0 is increased if g_0 is higher than a minimum value.

The same applies to the numerator of the second part of $G_{SBD}(\omega)$;

$$\begin{aligned} &\left[Im(D_{p0}(j\omega)(g_0+j\omega))\right]^2 = \\ &(Im(D_{p0}))^2 g_0^2 + 2(Im(D_{p0})Re(D_{p0}))\omega g_0 + (Re(D_{p0}))^2\omega^2 \end{aligned} \tag{2.32}$$

Note that this is a quadratic function of g_0 and $(ImD_{p0})^2 > 0$. Obviously for positive g_0, the value of this function is increased as g_0 is increased if g_0 is higher than a minimum value.

Note also that the denominator of the $G_{SBD}(\omega)$ is independent of g_0. Finally, it can be concluded that as g_0 is increased stability margin is also increased if g_0 is higher than a certain value called its minimum value.

$\square$

Example: Consider the following example;

$$P_n(s) = \frac{s+1}{s^2+2s+2} \tag{2.33}$$

$$P(s) = \frac{s+1}{s^2+(2+\Delta a_1)s+(2+\Delta a_0)} \tag{2.34}$$

$$Q(s) = \frac{g_0}{s+g_0} \tag{2.35}$$

Here;

$$\begin{aligned} P_{char,0}(s) = D_{p0}(s) + g_0 D_{p0}(s) = \\ 2g_0 + (2+2g_0)s + (2+g_0)s^2 + s^3 \end{aligned} \tag{2.36}$$

It is possible to draw r_{max} versus ω plot for different g_0 values. For example, the plot is depicted for $g_0 = 20$ in Figure 2.5. Here, $\rho = \inf_{\omega>0}(r_{max}(\omega)) \cong 21.04$ .The value sets

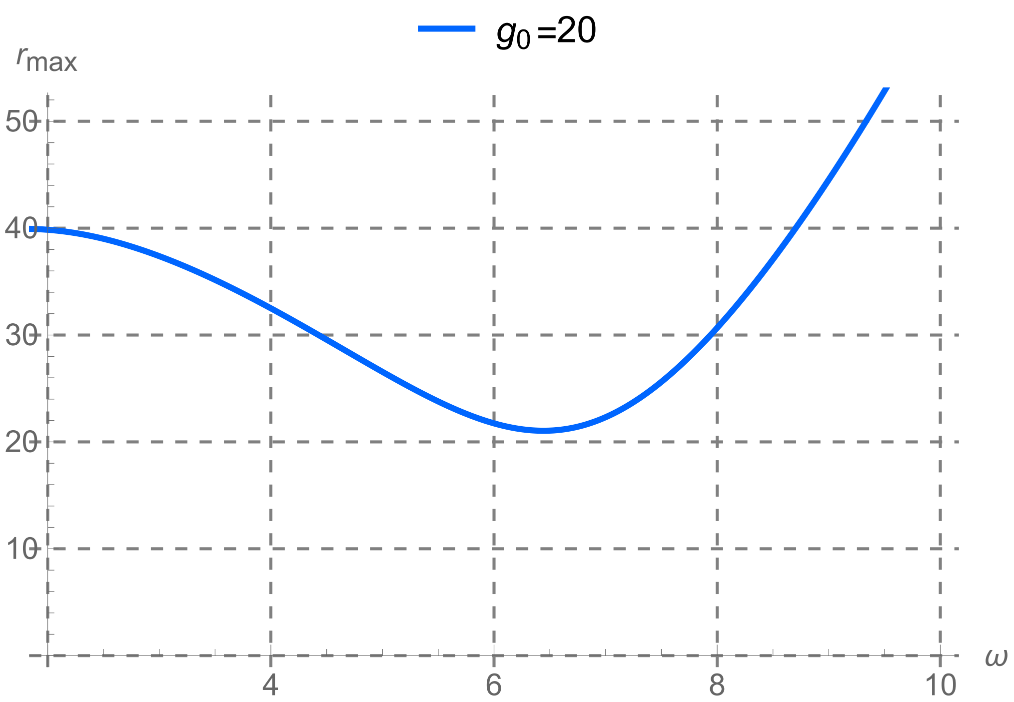

Figure 2.5 : $r_{max} - \omega$ plot for $g_0 = 20$.

for different frequencies $r = 1$ and $r = 21.04$ are given in Figure 2.6. $r_{max} - \omega$ plot for different g_0 values is given in Figure 2.7. As it can be observed from the value sets, although the centers of the ellipses are the same, major and minor axes of the ellipses change dramatically as with the value of r. This is a typical result for case-1 because of the following observation: Consider the ellipsoid inequality given in (2.37) where the $P_0(j\omega)$ represents the center of the ellipse and eigenvalues of W represent the lengths of the major and the minor axes of the ellipse.

$$P_0^T(j\omega)WP_0(j\omega) \leq r^2 \tag{2.37}$$

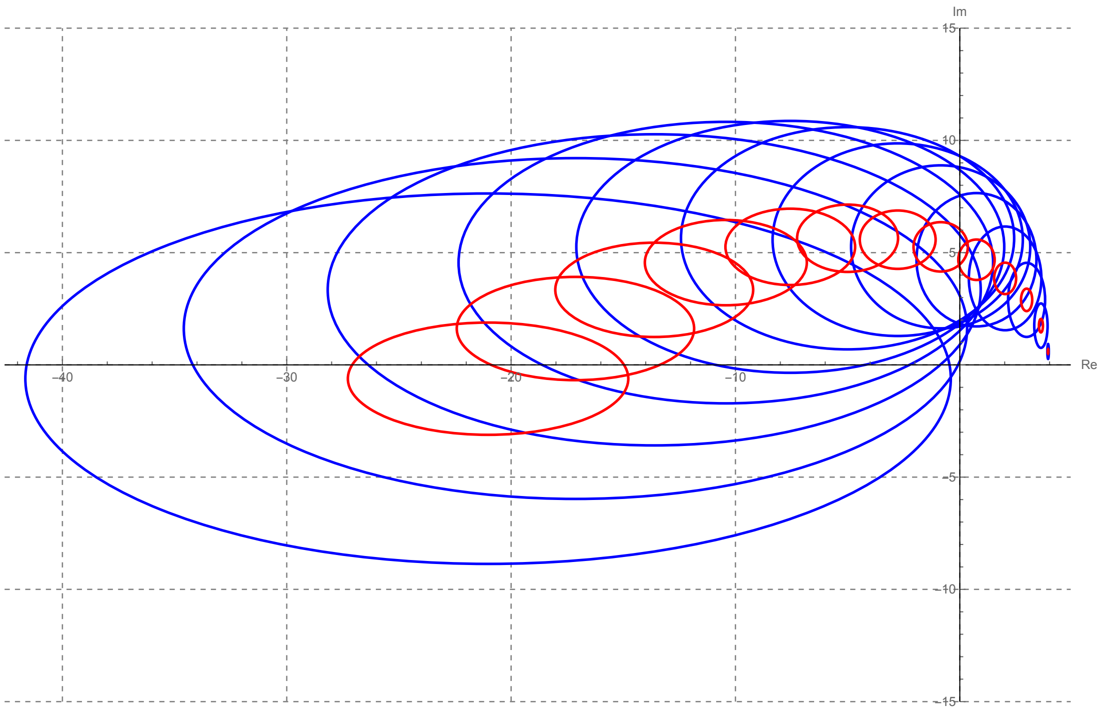

Figure 2.6 : Value set for $r = 1$ (Red), $r = 21.04$ (blue), $g_0 = 20$.

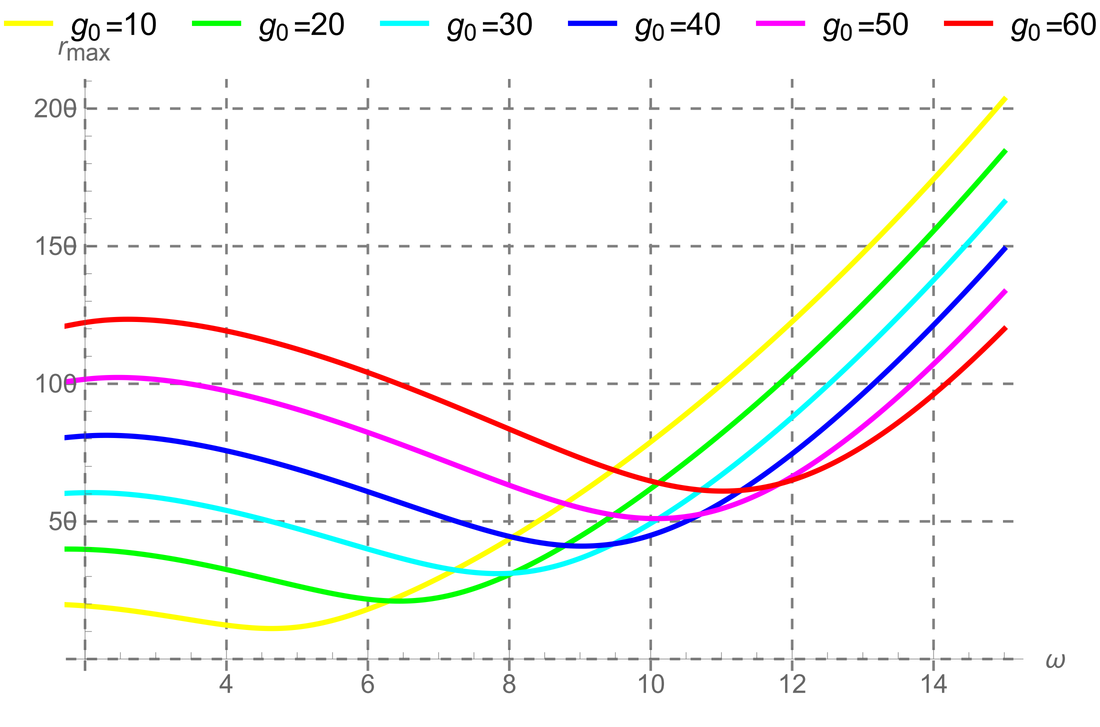

Figure 2.7 : $r_{max} - \omega$ plot for $g_0 = \{20, 30, 40, 50, 60\}$.

For case-1, the length of the major axis, (R_{major}), is $r\sqrt{max(\lambda(W^{-1}))}$ and W^{-1} is independent of g_0. However, the distance between the center of the ellipse and the origin, $|P_0|$ is $|D_n(j\omega)||N_n(j\omega)|\sqrt{g_0^2+\omega^2}$. Therefore, the axis length of ellipses does not depend on g_0, whereas the distance between the center of the ellipses and the origin increases as g_0 increases. This fact can also be observed in Figure 2.8, where value sets for different frequencies are depicted for $g_0 = 10$ and $g_0 = 15$.

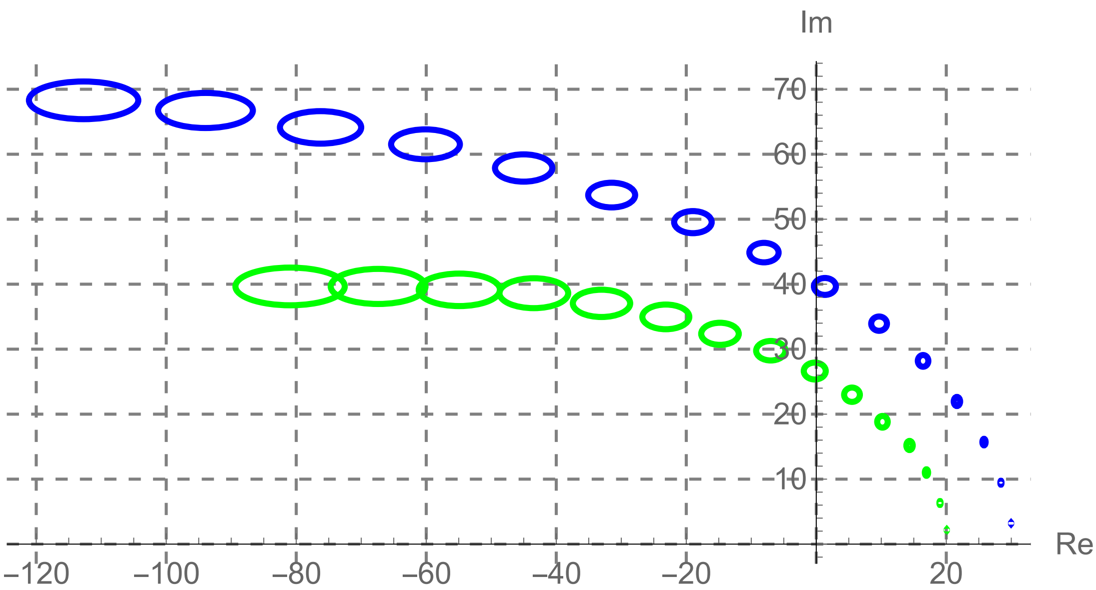

Figure 2.8 : Value set for $g_0 = 10$ (green) and $g_0 = 15$ (blue).

2.2.2 Case-2: Low-ordered P_n case

Theorem 2.1 gives a useful framework to investigate different cases of DOB based systems. The case where the nominal plant model has a more simpler structure than the uncertain plant is considered in this part. The following additional assumptions are given.

Assumption 3: Only the denominator of P is uncertain and relative degree of P is 1.

Assumption 4: $P_n(s)$ is a first order transfer function.

Lemma 2.2: *Consider the inner loop of DOB structure given in Figure 2.4 and the inner loop transfer function given in (2.16). Under Assumptions 2, 3 and 4, the robustness margin can be determined as a function of DOB bandwidth as follows,* $min\left\{\sqrt{G_{SBD}(\omega)}, |g_0 N_p(0)|\right\} > r$.

Proof: Note that, whatever the uncertain plant is, the nominal plant model is assumed to be of first order. This assumption has practical meaning especially when the exact plant model structure is not known. Here;

$$Q(s) = \frac{g_0}{g_0 + s} \tag{2.38}$$

$$\frac{N_p(s)}{D_p(s)} = \frac{s^k + b_{k-1}s^{k-1} + \cdots + b_0}{s^{k+1} + (a_k + \Delta a_k)s^k + \cdots + (a_0 + \Delta a_0)} \tag{2.39}$$

$$\frac{N_n(s)}{D_n(s)} = \frac{K}{\tau s + 1} \tag{2.40}$$

It is possible to factorize $D_p(s)$ as follows;

$$D_p(s) = D_{p0}(s) + D_{p\Delta}(s) \tag{2.41}$$

where;

$$D_{p0}(s) = s^{k+1} + a_k s^k + \ldots + a_0 = D_n(s) \tag{2.42}$$

$$D_{p\Delta}(s) = \Delta a_k s^k + \ldots + \Delta a_0 \tag{2.43}$$

Finally, the nominal characteristic polynomial is obtained as follows

$$P_{char,0}(s) = D_{p0}(s)Ks + g_0(\tau s + 1)N_p(s) \tag{2.44}$$

Now, according to Theorem 2.1, $D_{p0}(s)Ks + g_0(\tau s + 1)N_p(s)$ must be stable. Assumption 2 states that $D_p(s)$ is stable and also N_p is stable if the plant is minimum phase. However, the first condition of Theorem 2.1 is not held automatically. Q and P_n must be selected carefully. Now, the second condition of Theorem 2.1 requires that;

$$|g_0 b_0| > r \tag{2.45}$$

Obviously, as the bandwidth of the disturbance observer is increased, this condition is satisfied more easily. Finally, due to the last condition of Theorem 2.1, it is required that,

$$G_{SBD}(\omega) = \frac{[Re(P_{char,0}(j\omega))]^2}{\sum_{i_{even}} \omega^{2i}} + \frac{[Im(P_{char,0}(j\omega))]^2}{\sum_{i_{odd}} \omega^{2i}} > r^2 \tag{2.46}$$

Let;

$$D_{p0} = ReD_{p0} + jImD_{p0} \tag{2.47}$$

$$N_p = ReN_p + jImN_p \tag{2.48}$$

Then, the numerator of the first part of $G_{SBD}(\omega)$ is;

$$\begin{aligned} &\left[Re(P_{char,0}(j\omega))\right]^2 = \\ &g_0^2\left(Re(N_p) - \tau\omega Im(D_p)\right)^2 + 2g_0\omega K Im(D_{p0})\left(-Re(N_p) + \tau\omega Im(D_p)\right) \\ &+(Im(D_{p0}))^2 K^2 \omega^2 \end{aligned} \tag{2.49}$$

Note that this is a quadratic function of g_0 . When $g_0 > \frac{\omega K Im(D_{p0})}{Re(N_p)-\tau\omega Im(D_p)}$ and $g_0 > 0$, the value of this function increases as g_0 increases. The same applies to the numerator of the second part of $G_{SBD}(\omega)$;

$$\begin{aligned}\left[Im(P_{char,0}(j\omega))\right]^2 &= g_0^2\left(Im(D_p)+\tau\omega Re(N_p)\right)^2 \\ &+2g_0\omega K Re(D_{p0})\left(Im(D_p)+\tau\omega Re(N_p)\right) \\ &+(Re(D_{p0}))^2K^2\omega^2\end{aligned} \tag{2.50}$$

Note that this is a quadratic function of g_0 . When $g_0 > \frac{\omega K Re(D_{p0})}{Im(N_p)+\tau\omega Re(D_p)}$ and $g_0 > 0$, the value of this function increases as g_0 increases. Finally, it can be concluded that the robustness margin can be determined considering $min\left\{\sqrt{G_{SBD}(\omega)}, |g_0 N_p(0)|\right\} > r$.

$\square$

Although it is not guaranteed that $G_{SBD}(\omega)$ increases with g_0, this is a promising result that enables us the use a simpler structure for the filter design.

Example: Consider the previous example with a low-order P_n, where

$$P_n(s) = \frac{1}{s+2} \tag{2.51}$$

$$P(s) = \frac{s+1}{s^2+(2+\Delta a_1)s+(2+\Delta a_0)} \tag{2.52}$$

$$Q(s) = \frac{g_0}{s+g_0} \tag{2.53}$$

Then, the closed-system nominal characteristic polynomial is calculated as follows;

$$P_{char,0}(s) = s^3+(2+g_0)s^2+(2+2g_0)s+2g_0 \tag{2.54}$$

It is possible to draw r_{max} versus ω plots for different g_0 values as depicted in Figure 2.9. For $g_0 = 10$, $\rho = \inf_{\omega>0}(r_{max}(\omega)) \cong 0.33$. The value sets with different frequencies are depicted for $r = 0.1$ and $r = 0.33$ when $g = 10$ in Figure 2.10. As it can be observed from Figure 2.10, although the centers of the ellipses are the same, lengths of major and minor axes of ellipses change dramatically with r. Note that, it is found in previous example (Case-1) that $\rho = 21.04$. This example demonstrates also that poorly designed P_n lead to a week robustness margin. Increasing the time constant of the nominal plant improves the robustness margin. Consider the same plant with $P_n(s) = \frac{0.1}{s+0.2}$ and consider the DOB filter $Q(s) = \frac{1}{s+2}$, where the time constant of the Q filter is 5 times less than the nominal plant as in the previous example ($g_0 = 10$). It can be shown that $\rho \cong 1.93$ in this case. Figure 2.11 shows r_{max} versus ω plots for a practical range of g_0.

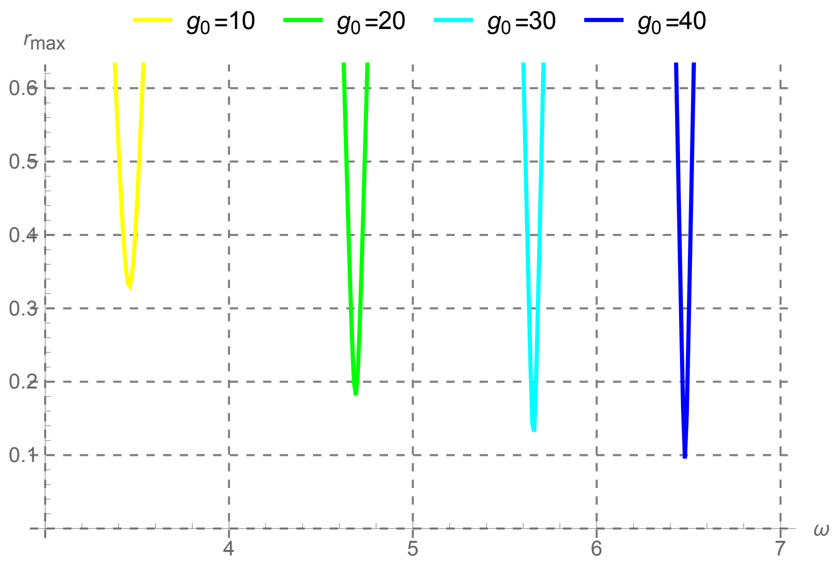

Figure 2.9 : $r_{max} - \omega$ plot for $g_0 = \{10, 20, 30, 40\}$.

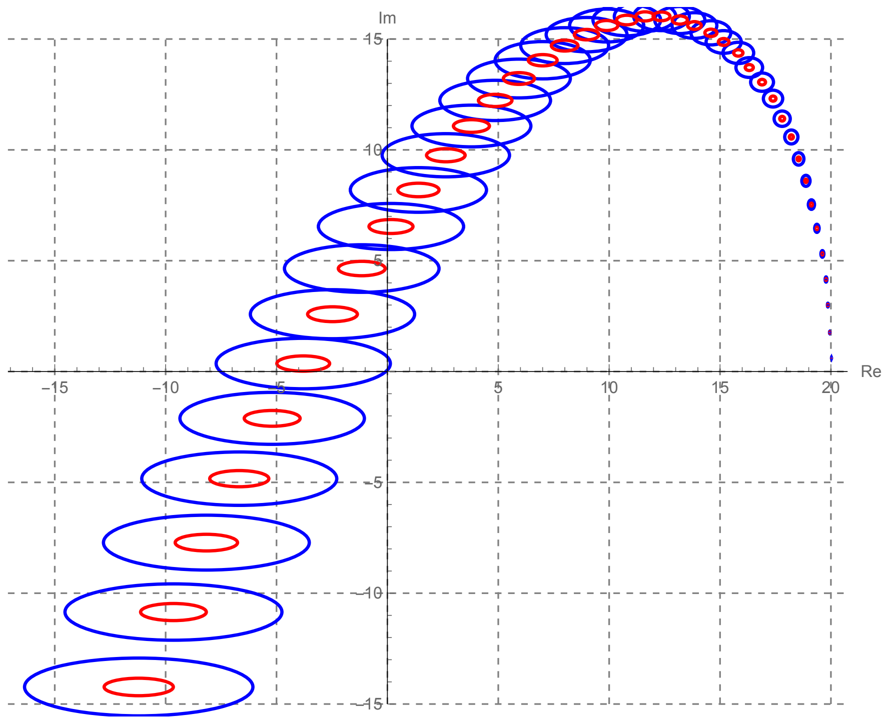

Figure 2.10 : Value sets for $r = 0.1$ (red), $r = 0.33$ (blue), $g_0 = 10$.

2.3 Affine Linear Polynomials

The previous section dealt with the case where the uncertain polynomial family is of interval type. In this case, Theorem 2.1 gives a refined solution to the problem of determining the robust stability radius of DOB based inner loop polynomial. However,

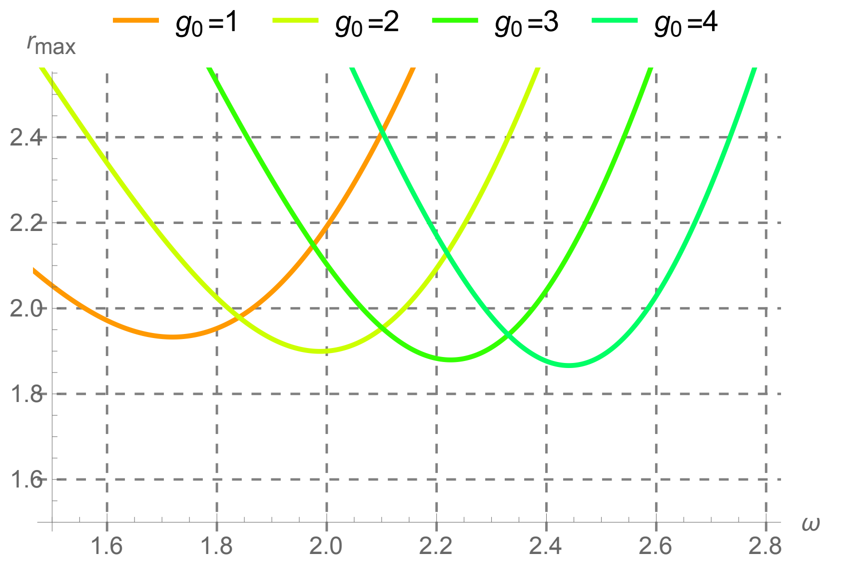

Figure 2.11 : $r_{max} - \omega$ plot for $g_0 = \{1, 2, 3, 4\}$.

in a more general case, the inner loop of a DOB based system results in an affine linear characteristic polynomial family. Unfortunately, the available machinery seems not very suitable for such disturbance observer-based systems in the first place.

This problem is already considered in [79], however, the proposed machinery includes matrix inversions which may lead to unnecessarily complex solutions. The analytically complex nature of the approach given in [79] directs the interest to the technique given in [80], which has more geometric intuition. The reader may refer to [81] for a detailed discussion. However, a refined solution that shows the analytical relation between the bandwidth of the DOB and the stability radius in parameter space has not been achieved yet.

In [34], the problem is considered only for the uncertainty represented by l_∞ norm (box representation). However, the relation between the bandwidth of DOB and the robustness margin has not been given explicitly. The aim of this section is to provide a refined approach to analyze the robustness margin and the bandwidth of DOB where the uncertainty is present in both numerator and denominator of the plant.

2.3.1 Minimum-phase plant case

The following theorem can be expressed if the plant transfer function is minimum phase.

Theorem 2.2: *Consider the uncertain plant with disturbance observer described by;*

$$P(s) = \frac{N_p(s)}{D_p(s)} \tag{2.55}$$

$$P_n(s) = \frac{N_n(s)}{D_n(s)} \tag{2.56}$$

$$Q(s) = \frac{N_q(s)}{D_q(s)} \tag{2.57}$$

with uncertainty bounding set $\|q\|_2 \leq r$. Assume that;

1) $P(s)$ is minimum phase and Relative degree of $P(s)$ is one, that is, $n-m=1$

2) The Nominal plant P_n is stable

Then, the relation between the DOB bandwidth (g_0) and the robustness margin is expressed as follows;

$$P_0^T\,(j\omega)\,\frac{W}{r^2}\,P_0\,(j\omega)>1 \tag{2.58}$$

where W satisfies the following equation;

$$W^{-1}r^2{=}\left(M_1{+}g_0^2M_2\right)r^2 \tag{2.59}$$

in which M_1 and M_2 are positive definite matrices.

Proof: The proof follows the same reasoning given in [77].

$$N_p\,(s) \;=\; \sum_{i=0}^{m} b_i s^i + \sum_{i=0}^{m} \Delta b_i s^i N_p\,(s) = N_n\,(s) + N_\Delta(s) \tag{2.60}$$

$$D_p\,(s) \;=\; \sum_{i=0}^{n} a_i s^i + \sum_{i=0}^{m} \Delta a_i s^i = D_n\,(s) + D_\Delta(s) \tag{2.61}$$

$$N_\Delta\,(j\omega) \;\triangleq\; \sum_{i=0}^{m} \Delta b_i (j\omega)^i \tag{2.62}$$

$$D_\Delta\,(j\omega) \;\triangleq\; \sum_{i=0}^{n} \Delta a_i (j\omega)^i \tag{2.63}$$

The closed-loop system characteristic polynomial $P_c(s)$ is;

$$P_c\,(s) \;=\; D_p N_n\left(D_q - N_q\right) + D_n N_p N_q \tag{2.64}$$

$$P_c\,(s) \;=\; D_n N_n\,(s+g_0) + D_\Delta N_n s + N_\Delta D_n g_0 = P_0\,(s) + P_\Delta\,(s) \tag{2.65}$$

$$P_0\,(s) \;\triangleq\; D_n N_n\,(s+g_0) \tag{2.66}$$

$$P_\Delta\,(s) \;\triangleq\; D_\Delta N_n s + N_\Delta D_n g_0 \tag{2.67}$$

(2.68)

and

$$P_c(j\omega,Q)=\{P_c(j\omega,q):q\in Q\} \tag{2.69}$$

where;

$$q\triangleq(\Delta a_i,\Delta b_k)\,,\; i=0,1\dots n \; and \; k=0,1,\dots,m \tag{2.70}$$

$P_0(j\omega)$ can be written using the following expressions;

$$\begin{aligned} Im(N_n) &\triangleq Im\{N_n(j\omega)\} && (2.71)\\ Re(N_n) &\triangleq Re\{N_n(j\omega)\} && (2.72)\\ Im(D_n) &\triangleq Im\{D_n(j\omega)\} && (2.73)\\ Re(D_n) &\triangleq Re\{D_n(j\omega)\} && (2.74)\end{aligned}$$

$$D_\Delta(j\omega)N_n(j\omega)(j\omega)=\sum_{i=0}^{n}\Delta a_i N_n(j\omega)(j\omega)^{i+1} \tag{2.75}$$

$$\begin{aligned} &= \begin{pmatrix} Re(N_n) & -Im(N_n)\\ Im(N_n) & Re(N_n)\end{pmatrix}\begin{pmatrix} 0 & -\omega^2 & \dots\\ \omega & 0 & \dots\end{pmatrix}\begin{pmatrix}\Delta a_0\\ \vdots\\ \Delta a_n\end{pmatrix} && (2.76)\\ &= NA_n\Delta a && (2.77)\end{aligned}$$

$$N_\Delta(j\omega)D_n(j\omega)g_0=\sum_{i=0}^{m}g_0\Delta b_i D_n(j\omega)(j\omega)^i \tag{2.78}$$

$$\begin{aligned} &= g_0\begin{pmatrix} Re(D_n) & -Im(D_n)\\ Im(D_n) & Re(D_n)\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & \omega\\ -\omega^2 & 0\\ \dots & \dots\end{pmatrix}^T\begin{pmatrix}\Delta b_0\\ \vdots\\ \Delta b_m\end{pmatrix} && (2.79)\\ &= g_0DA_d\Delta b && (2.80)\end{aligned}$$

where

$$D \triangleq \begin{pmatrix} Re(D_n) & -Im(D_n) \\ Im(D_n) & Re(D_n) \end{pmatrix} \tag{2.81}$$

$$N \triangleq \begin{pmatrix} Re(N_n) & -Im(N_n) \\ Im(N_n) & Re(N_n) \end{pmatrix} \tag{2.82}$$

$$A_d \triangleq \begin{pmatrix} 1 & 0 & -\omega^2 & 0 & \dots \\ 0 & \omega & 0 & -\omega^3 & \dots \end{pmatrix} \tag{2.83}$$

$$A_n \triangleq \begin{pmatrix} 0 & -\omega^2 & 0 & \dots \\ \omega & 0 & -\omega^3 & \dots \end{pmatrix} \tag{2.84}$$

$$\Delta a \triangleq \begin{pmatrix} \Delta a_0 \\ \vdots \\ \Delta a_n \end{pmatrix} \tag{2.85}$$

$$\Delta b \triangleq \begin{pmatrix} \Delta b_0 \\ \vdots \\ \Delta b_n \end{pmatrix} \tag{2.86}$$

$$\Omega \triangleq \begin{pmatrix} N & g_0 D \end{pmatrix} \tag{2.87}$$

$$A \triangleq \begin{pmatrix} A_n & 0 \\ 0 & A_d \end{pmatrix} \tag{2.88}$$

$$q \triangleq \begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix} \tag{2.89}$$

For a fixed $\omega > 0$

$$z \in P_c\,(j\omega, Q) \iff \tag{2.90}$$

$$z = P_0\,(j\omega) + \begin{pmatrix} N & g_0 D \end{pmatrix} \begin{pmatrix} A_n & 0 \\ 0 & A_d \end{pmatrix} \begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix} = P_0\,(j\omega) + \Omega\, A\, q \tag{2.91}$$

For $q \in Q$, any minimum norm solution q^{min} of the z satisfies $\left\|q^{min}\right\|_2 \leq r$. At this point, it is time to consider the system of equations with infinitely many solutions, such as $z = P_0\,(j\omega) + \Omega\, A\, q$. It is convenient to find the smallest norm solving the optimization problem $minimize_{x \in \mathbb{R}^n}\, \|x\|\ \ subject\ to\ Ax = b$.

Lemma 2.3 [82]: *A vector x^{min} satisfying $Ax^{min} = b$ is the minimum norm solution of the system of equations $Ax = b$ if, and only if, it can be written as $x^{min} = A^T v$ for some v.*

So, the minimum norm solution x^{min} can be found by solving the system $AA^T v = b$ for v, then setting $x^{min} = A^T v$.

Thus, considering the $z = P_0\,(j\omega) + \Omega\,A\,q$;

$$\begin{aligned} z - P_0\,(j\omega) &= (\Omega A)\,q && (2.92)\\ q^{min} &= (\Omega A)^T v && (2.93)\\ z{-}P_0\,(j\omega) &= (\Omega A)\,(\Omega A)^T v && (2.94)\\ v &= \left[(\Omega A)\,(\Omega A)^T\right]^{-1}(z{-}P_0(j\omega)) && (2.95)\\ W^{-1} &\triangleq (\Omega A)\,(\Omega A)^T && (2.96)\\ q^{min} &= (\Omega A)^T W(z{-}P_0(j\omega)) && (2.97) \end{aligned}$$

Then,

$$\left\|q^{min}\right\|_2 \leq r \qquad (2.98)$$

$$(z{-}P_0(j\omega)^T)W(z{-}P_0(j\omega)){=}(q^{min})^2{<}r^2 \qquad (2.99)$$

At this point, the rank condition must be checked in order to guarantee that the minimum norm solution is unique. To this end Rank (ΩA) must be 2. Considering A_n and A_d, $Rank(A) = 4$ and also, since D_n and N_n are assumed to be coprime, $Det\,(N) \neq 0$ or $Det(D) \neq 0$. This guarantees that Rank $(\Omega A) = 2$. In view of the ellipsoid expression, it is possible to check whether the origin is included or not. In this case, the requirement is;

$$P_0^T\,(j\omega)\,W P_0\,(j\omega) {>} r^2 \qquad (2.100)$$

and considering the value set, the larger $P_0(j\omega)$ and W, the larger the robustness margin is. However, W and g_0 are inversely proportional, that is;

$$\begin{aligned} W^{-1} &= (\Omega A)\,(\Omega A)^T && (2.101)\\ &= (NA_nA_n^TN^T{+}g_0^2DA_dA_d^TD^T) && (2.102) \end{aligned}$$

As g_0 is increased, W gets smaller (W^{-1} gets larger). To show this relation clearly it is possible to use the eigenvalues of W.

Let

$$\begin{aligned} M_1 &\triangleq NA_nA_n^TN^T && (2.103)\\ M_2 &\triangleq DA_dA_d^TD^T && (2.104)\\ M &\triangleq M_1{+}g_0^2M_2 && (2.105) \end{aligned}$$

$A_nA_n^T$ and $A_dA_d^T$ are diagonal matrices of the form;

$$A_nA_n^T = \begin{pmatrix} \sum_{i=2,4,..2k,} \omega^{2i} & 0 \\ 0 & \sum_{i=1,3,..2k+1} \omega^{2i} \end{pmatrix} \tag{2.106}$$

$$\triangleq \begin{pmatrix} A_{n1} & 0 \\ 0 & A_{n2} \end{pmatrix} \tag{2.107}$$

$$A_dA_d^T = \begin{pmatrix} \sum_{i=0,2,4,..2k} \omega^{2i} & 0 \\ 0 & \sum_{i=1,3,..2k+1} \omega^{2i} \end{pmatrix} \tag{2.108}$$

$$\triangleq \begin{pmatrix} A_{d1} & 0 \\ 0 & A_{d2} \end{pmatrix} \tag{2.109}$$

Finally, the relation between the DOB bandwidth (g_0) and the robustness margin is expressed as follows;

$$P_0^T(j\omega) W P_0(j\omega) > r^2 \tag{2.110}$$

$$P_0^T(j\omega) \frac{W}{r^2} P_0(j\omega) > 1 \tag{2.111}$$

$$W^{-1}r^2 = \left(M_1 + g_0^2 M_2\right) r^2 \tag{2.112}$$

□

Corollary 4.3: *It is not guaranteed that the robustness margin of the system is increased as the bandwidth of the DOB is increased in general.*

Proof: Let us define;

$$|N_n| \triangleq \sqrt{(Re(N_n)(j\omega))^2 + (Im(N_n)(j\omega))^2} \tag{2.113}$$

$$|D_n| \triangleq \sqrt{(Re(D_n)(j\omega))^2 + (Im(D_n)(j\omega))^2} \tag{2.114}$$

Since finding the eigenvalues of W is much more complicated compared to that of W^{-1}, the eigenvalues of W^{-1} are considered in the rest of the proof. Then;

$$\lambda(M_1)_1 = A_{n1}|N_n|^2 \tag{2.115}$$

$$\lambda(M_1)_2 = A_{n2}|N_n|^2 \tag{2.116}$$

$$\lambda(M_2)_1 = A_{d1}|D_n|^2 \tag{2.117}$$

$$\lambda(M_2)_2 = A_{d2}|D_n|^2 \tag{2.118}$$

Lemma 2.4 [83]: *Let A and B be real symmetric matrices. Then A, B and $A+B$ have real eigenvalues. Let $\lambda^+(.)$ and $\lambda^-(.)$ represent maximum and minimum eigenvalues* *of (.) respectively. Then;*

$$\lambda^{-}(A)+\lambda^{-}(B)\leq\lambda^{-}(A+B)\leq\lambda^{+}(A+B)\leq\lambda^{+}(A)+\lambda^{+}(B) \tag{2.119}$$

Major and minor axis lengths of the ellipsoid are $r\sqrt{\lambda^{+}(M)}$ and $r\sqrt{\lambda^{-}(M))}$ respectively. Since the maximum eigenvalue of M is point of interest due to the major axis of the ellipsoid, it is possible to write the following inequality by using Lemma 2.4;

$$\lambda^{+}(M)<\lambda^{+}(M_1)+g_0^2\lambda^{+}(M_2) \tag{2.120}$$

At this stage, the relation between the major axis of the ellipsoid and the bandwidth of the DOB (g_0) is expressed as;

$$\left|R_{major}\right|=r\sqrt{\lambda^{+}(M_1)+g_0^2\lambda^{+}(M_2)} \tag{2.121}$$

Clearly, as the g_0 is increased, the major axis of the ellipsoid is increased with the order of g_0. Now, consider the distance between the center of the ellipsoid and the origin (Figure 2.12).

$$\begin{aligned} P_0(j\omega) &= D_n(j\omega)N_n(j\omega)(j\omega+g_0) && (2.122)\\ |P_0(j\omega)| &= |D_n|\,|N_n|\sqrt{g_0^2+\omega^2} && (2.123) \end{aligned}$$

As the g_0 is increased, the distance between ellipsoid and the origin is also increased with the order of g_0.

Since the increment of $|P_0(j\omega)|$ and increment of $\left|R_{major}\right|$ with g_0 depend on the $N_n(j\omega)$ and $A_{d1,d2}$, it can be concluded that; it is not guaranteed that the robustness margin of the system is increased with the bandwidth of the DOB in general.

□

Example: Consider the following plant and Q filter ;

$$\begin{aligned} P(s) &= \frac{(1+\Delta b_1)s+(\Delta b_0+5)}{(3+\Delta a_2)s^2+(\Delta a_1+4)s+\Delta a_0+6} && (2.124)\\ P_n(s) &= \frac{s+5}{3s^2+4s+6} && (2.125)\\ Q(s) &= \frac{g_0}{s+g_0} && (2.126) \end{aligned}$$

then the characteristic equation becomes;

$$\begin{gathered} P_c(s,q)=P_0(s)+\Delta a_0\,P_1(s)+\Delta a_1P_2(s)\\ +\Delta a_2\,P_3(s)+\Delta b_0P_4(s)+\Delta b_1P_5(s) \end{gathered} \tag{2.127}$$

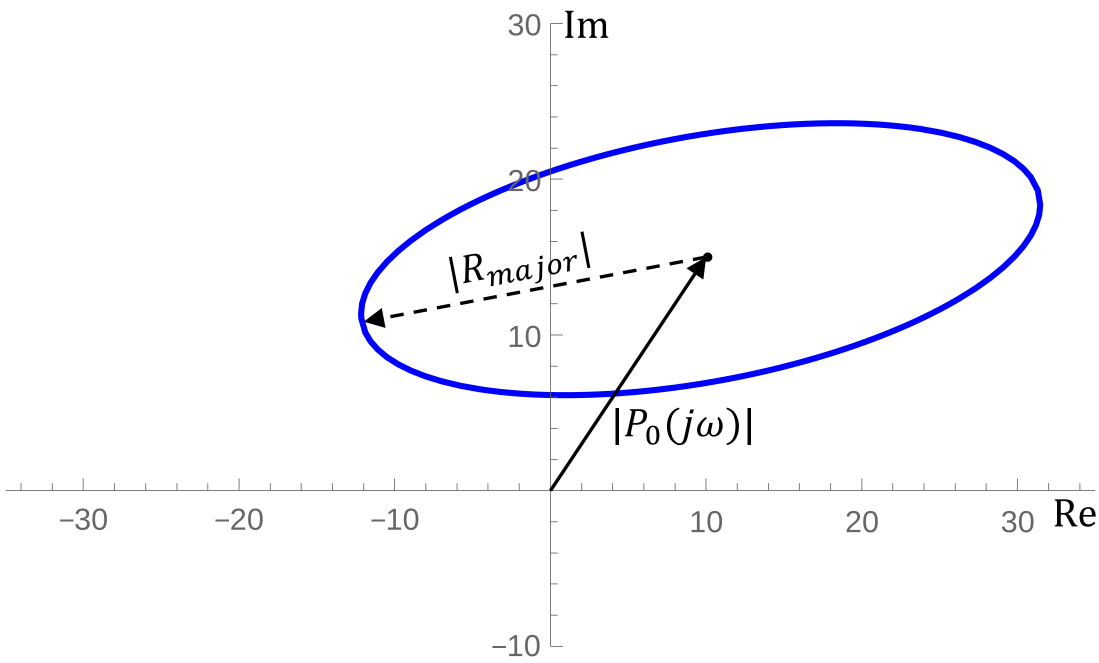

Figure 2.12 : Representation of the $|P_0(j\omega)|$ and $|R_{major}|$.

where;

$$P_0(s) = 3s^4 + (19+3g)s^3 + (26+19g)s^2 + (30+26g)s + 30g \tag{2.128}$$

$$P_1(s) = 5s + s^2 \tag{2.129}$$

$$P_2(s) = 5s^2 + s^3 \tag{2.130}$$

$$P_3(s) = 5s^3 + s^4 \tag{2.131}$$

$$P_4(s) = g_0(6 + 4s + 3s^2) \tag{2.132}$$

$$P_5(s) = g_0(6s + 4s^2 + 3s^3) \tag{2.133}$$

$$\tag{2.134}$$

then;

$$\Omega A = \begin{pmatrix} ReP_1 & ReP_2 & ReP_3 & ReP_4 & ReP_5 \\ ImP_1 & ImP_2 & ImP_3 & ImP_4 & ImP_5 \end{pmatrix} \tag{2.135}$$

$$W = \left((\Omega A)(\Omega A)^T\right)^{-1} \tag{2.136}$$

DOB based inner loop is robustly stable if,

- $P_0(s)$ is stable
- zero frequency condition $|P(j0)| > r$ is satisfied

and

$$P_0^T(j\omega)WP_0(j\omega) > r^2 \tag{2.137}$$

for all frequencies. Since $P_0(s)$ is stable and $|P_0(j0)| = 30g$, first two conditions can be satisfied easily. In order to find the minimum r value depending on the g_0, $P_0^T(j\omega)WP_0(j\omega)$ is plotted for various g_0 in Figure 2.13. Since $\rho =$

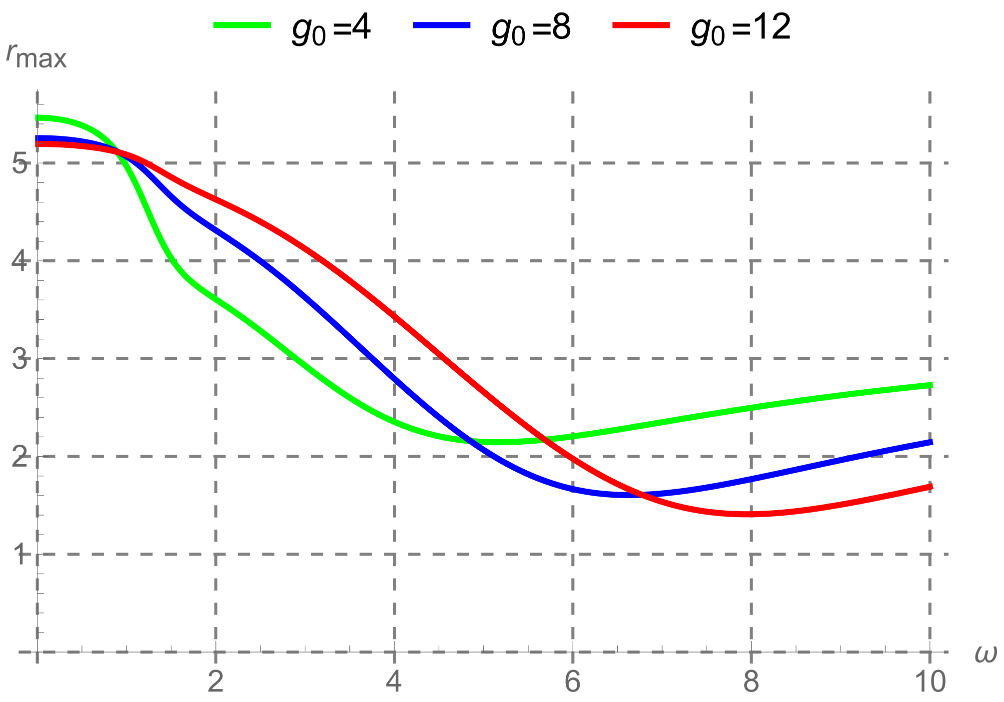

Figure 2.13 : $\sqrt{P_0^T(j\omega)WP_0(j\omega)}$ for different g_0. The minimum value of each graph represents the robustness margin of the system for a given g_0.

$min\{|30g_0|, |3|, r_{max}\}$, the robustness margin is determined by r_{max} and as the bandwidth of DOB is increased robustness margin decreases in the low-frequency range. Now, consider the same example with a different plant numerator, as follows,

$$P_{new}(s) = \frac{s + (\Delta b_0 + 5)}{(3+\Delta a_2)s^2 + (\Delta a_1 + 4)s + \Delta a_0 + 6} \tag{2.138}$$

where, the uncertain parameter, b_1, is omitted. In this case, the characteristic equation $P_c(s,q)$ becomes;

$$P_c(s,q) = P_0(s) + \Delta a_0\, P_1(s) + \Delta a_1 P_2(s) + \Delta a_2\, P_3(s) + \Delta b_0 P_4(s) \tag{2.139}$$

where, $P_1(s), P_2(s), P_3(s)$, and $P_4(s)$ are the same as in the previous example. It can be observed from Figure 2.14 that as g_0 is increased, the minimum value of

$P_0^T(j\omega)WP_0(j\omega)$ is also increased. That means that the uncertainty bound r is increased with the bandwidth of the disturbance observer until the $r_{max} = |a_n|$, since $\rho = min\{|30g_0|, |3|, r_{max}\}$. However, as it is seen in Figure 2.13, it is not always true that the robustness margin improves as with the g_0 considering the practical bandwidth design for DOB. Because of the noise concerns, DOB bandwidth has an upper bound in practical applications, therefore, throughout the study, DOB filter bandwidths have been selected such that, the bandwidth of DOB is several times larger than the bandwidth of the nominal plant. Actually, what the DOB filter does is to make

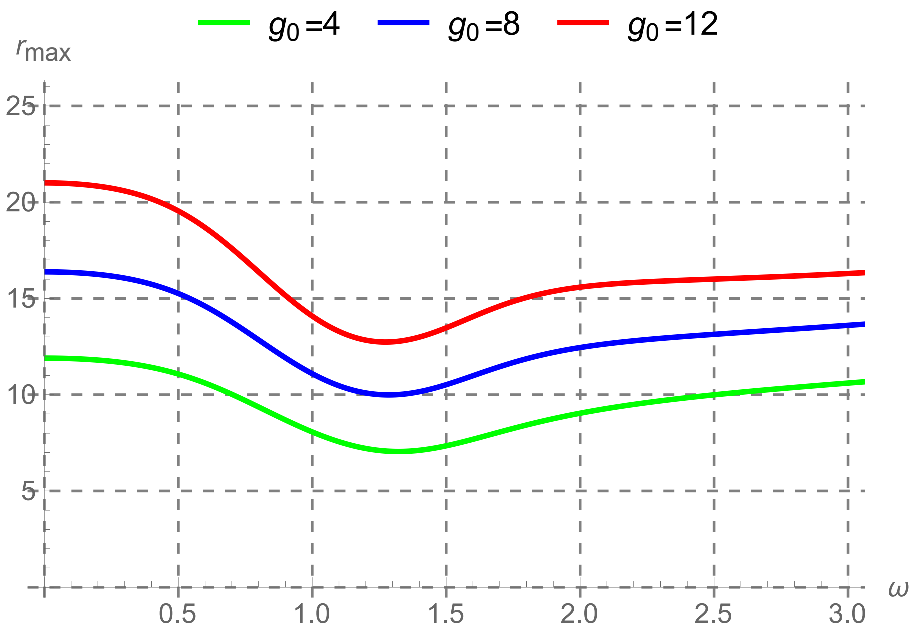

Figure 2.14 : $\sqrt{P_0^T(j\omega)WP_0(j\omega)}$ for different g_0. The minimum value of each graph represents the robustness margin of the system for a given g_0.

the inner loop dynamic response as close to that of the nominal plant $P_n(s)$ as possible. From this point of view, as the bandwidth of the DOB filter is increased, the inner loop poles get close to the nominal poles so that the robustness margin improves. As it can be seen from Figure 2.14, although increasing the bandwidth of DOB improves r_{max}, the radius of the uncertainty ball cannot be greater than $a_n = 3$. However, increasing the DOB bandwidth, the inner loop pole spread converges to the nominal poles. Figure 2.15 emphasizes this feature.

2.3.2 Non-minimum phase plant case

The disturbance observer uses the system inverse as it is shown in Figure 2.16 . This brings about an important problem, that is, if the plant is a non-minimum phase plant,

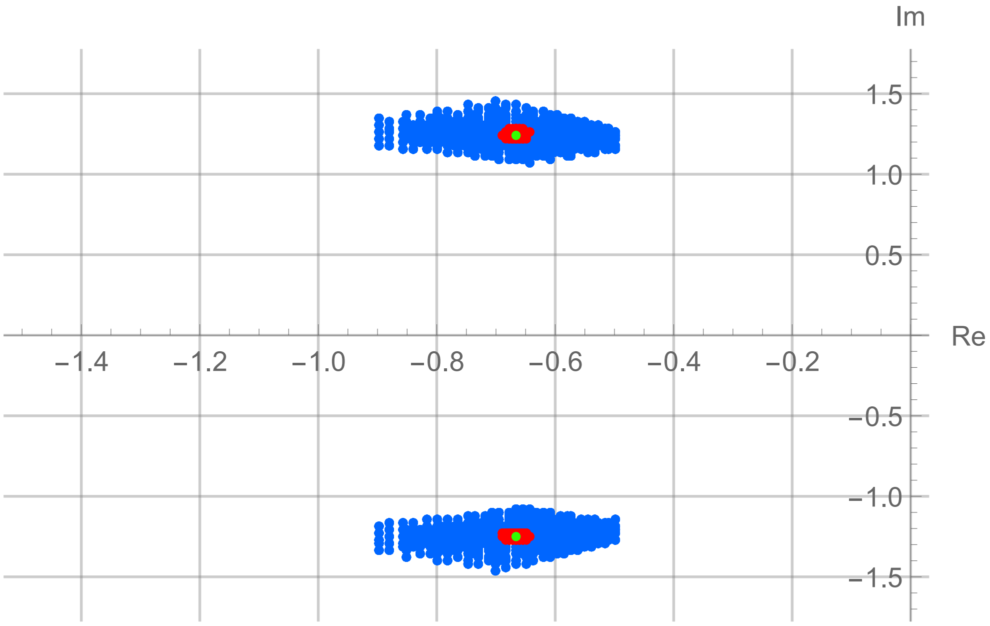

Figure 2.15 : Nominal poles (green) , pole spread of $P(s)$ (blue) and inner loop with $g_0 = 12$ (red).

internal stability problems occur. In this case, SBD theorem is not applicable.

The common approach for such a case is to divide the plant into two parts so that the non-minimum phase and the minimum phase parts are handled separately [20].

Lemma 2.5: *Theorem 2.2 can be utilized for plants with non-minimum phase zeros, if,*

1) Only the denominator of P is uncertain and relative degree of P is 1.

2) $P_n(s)$ is stable.

3) P(s) has a non-minimum phase zero.

4) $P_n(s)$ is factorized as, $P_n(s) = P_{n1}(s)\, P_{n2}(s)$

Where P_{n1} includes a pole at the reflection of right half plane zero and P_{n2} includes a right half plane zero.

Proof: By doing such a distinction, the invertible part of P_n may be used as a pre-filter as seen in Figure 2.16. For simplicity, the time delay is ignored. Considering the new assumptions following transfer functions are used in the analysis;

$$\frac{N_p(s)}{D_p(s)} = \frac{(s+b_k)(s+b_{k-1})\dots(-s+b_0)}{s^{k+1}+(a_k+\Delta a_k)s^k+\dots+(a_0+\Delta a_0)} \tag{2.140}$$

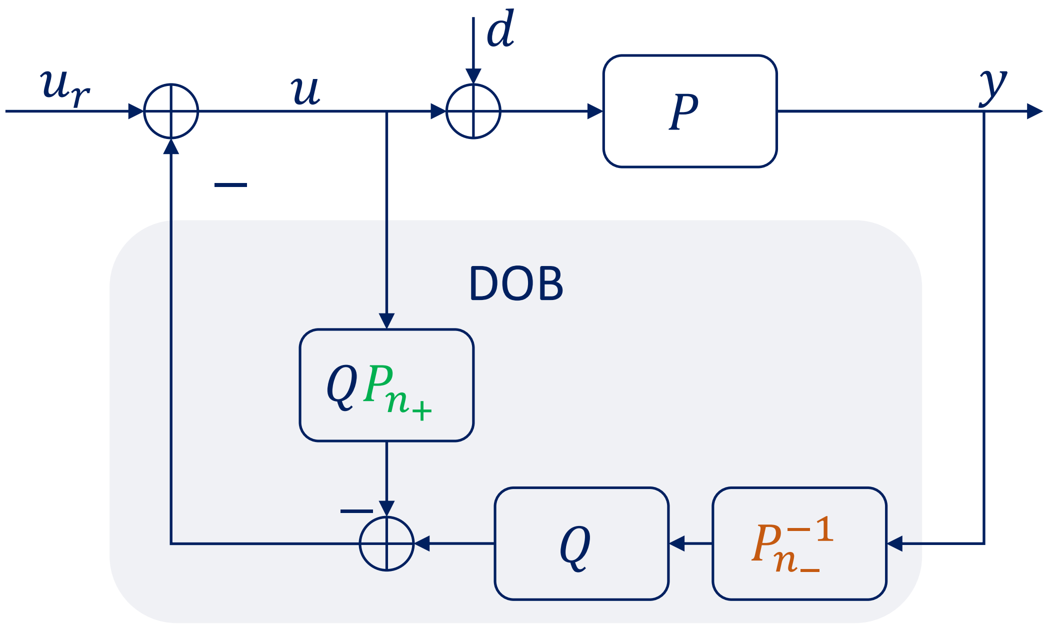

Figure 2.16 : DOB design for non-minimum phase systems.

$$P_{n1}(s) = \frac{N_{n1}(s)}{D_{n1}(s)} = \frac{(s+b_k)(s+b_{k-1})\ldots(s+b_0)}{s^{k+1}+a_k s^k+\cdots+a_0} \tag{2.141}$$

$$P_{n2}(s) = \frac{N_{n2}(s)}{D_{n2}(s)} = \frac{-s+b_0}{s+b_0} \tag{2.142}$$

$$Q(s) = \frac{g_0}{s+g_0} \tag{2.143}$$

The affine linear characteristic polynomial is given as follows;

$$P_{char}(s) = g_0 D_{p0}(s) N_{n2}(s) + D_p(s)[g_0(N_{n1}(s)-N_{n2}(s))+sN_{n1}(s)] \tag{2.144}$$

The nominal characteristic polynomial is obtained as follows;

$$P_{char,0}(s) = D_{p0}(s)N_{n1}(s)D_q(s) \tag{2.145}$$

Now, in this case;

$$P_0(s) \triangleq D_n N_{n1}(s+g_0) \tag{2.146}$$

$$P_\Delta(s) \triangleq D_\Delta N_{n1} s + D_\Delta(N_{n1}-N_{n2})g_0 \tag{2.147}$$

$$P_c(s) = P_0(s) + P_\Delta(s) \tag{2.148}$$

Then; for a fixed $\omega > 0$

$$z \in P_c(j\omega, Q) \iff \tag{2.149}$$

$$z = P_0(j\omega) + [N_{n1} + g_0(N_{n1}-N_{n2})]A_n \Delta a \tag{2.150}$$

$$= P_0(j\omega) + \Omega\, A\, q \tag{2.151}$$

where

$$\Omega \triangleq [N_{n1} + g_0(N_{n1} - N_{n2})] \tag{2.152}$$

$$A \triangleq A_n \tag{2.153}$$

$$q \triangleq \Delta a \tag{2.154}$$

since

$$W^{-1} = (\Omega A)(\Omega A)^T \tag{2.155}$$

$$= (\Omega A_n A_n^T \Omega^T) \tag{2.156}$$

$$\lambda(\Omega)_1 = A_{n1}|\Omega|^2 \tag{2.157}$$

$$\lambda(\Omega)_2 = A_{n2}|\Omega|^2 \tag{2.158}$$

it can be concluded that as the g_0 is increased, major axis of the ellipsoid is increased with the order of g_0. Finally;

$$P_0(j\omega) = D_n(j\omega)N_n(j\omega)(j\omega + g_0) \tag{2.159}$$

$$|P_0(j\omega)| = |D_n||N_n|\sqrt{g_0^2 + \omega^2} \tag{2.160}$$

As the g_0 is increased, the distance between the ellipsoid and the origin is also increased with the order of g_0.

Since P_0 and W are obtained, Theorem 2.2 can be directly applied.

□

Example: Consider the pan-tilt system transfer function given in [84];

$$P(s) = \frac{-s + 3101}{s + 3101}\frac{0.85308(s + 3101)}{s^2 + 369.8s + 1057} \tag{2.161}$$

Here, the following decomposition is made in order to split the plant into its minimum and non-minimum phase.

$$P_{n1}(s) = \frac{0.85308(s + 3101)}{s^2 + 369.8s + 1057} \tag{2.162}$$

$$P_{n2}(s) = \frac{-s + 3101}{s + 3101} \tag{2.163}$$

Finally, the nominal characteristic polynomial is as follows;

$$\begin{aligned} P_{char,0}(s) = g\left(1.146s^3 + 3069.53s^2 + 9794s + 2.796 \times 10^6\right) \\ +s\left(s^3 + 3470.8s^2 + 1.1478 \times 10^6 s + 3.2777 \times 10^6\right) \end{aligned} \tag{2.164}$$

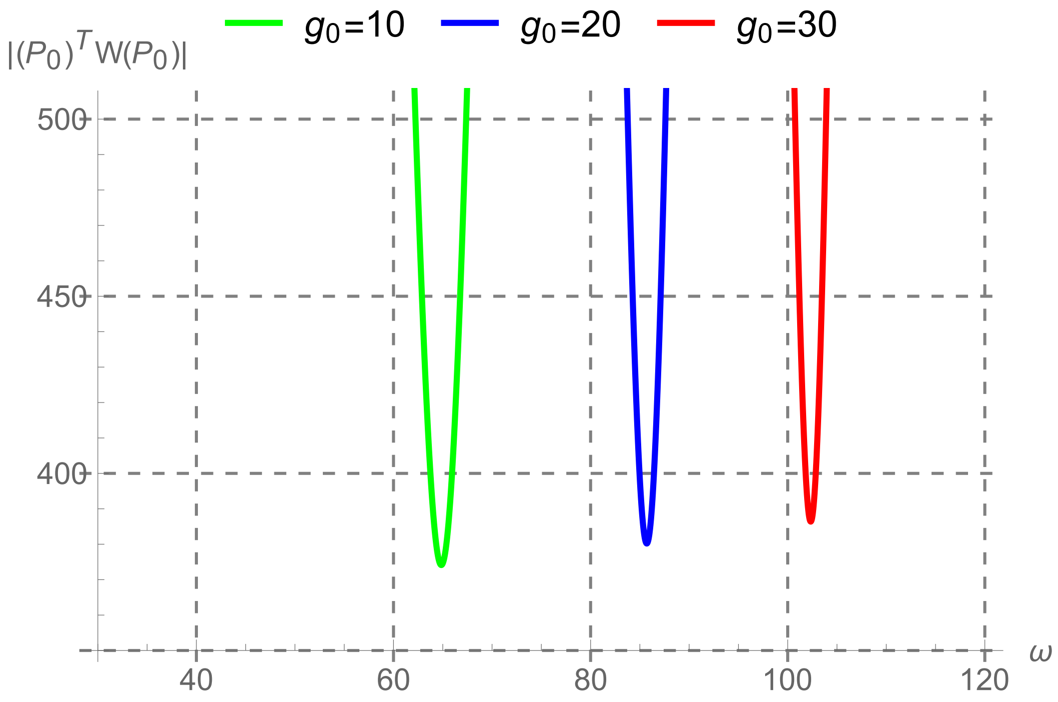

Figure 2.17 : $r_{max} - \omega$ plot for $g_0 = (10, 20, 30)$, non-minimum Phase Case.

It can be observed in Figure 2.17 that as g_0 increases, the robustness margin also increases. The robustness margin for a given DOB bandwidth can always be found by using the value set approach. For example, consider $g_0 = 10$, then r_{max} is 374.179. A zero inclusion occurs when $\|q\|_2 \geq 374.179$.

2.4 Conclusion

This study proposes a new way of analyzing the DOB based system utilizing the spherical polynomial approach. The value set concept for spherical polynomials has been adopted to validate the results. The main contributions are listed below;

- The spherical value set approach for uncertain polynomials is adopted for the first time for disturbance observer based control systems.
- The robustness margin for a given DOB based system has been systematically stated for the first time in the context of spherical polynomial families.
- Non-minimum phase case is examined, and bandwidth constraints are discussed.
- Effects of low-order DOB filter design are discussed.

The analyses show that for the case where the nominal and uncertain plants have the same structure and uncertain parameters present only on the denominator of the plant, then robustness margin improves as with the DOB filter bandwidth (if DOB bandwidth is higher than its minimum value). It is also shown that if the nominal plant is first-order regardless of the order of the uncertain plant, the robustness margin-DOB bandwidth relation is not straightforward and it is not guaranteed that the increased DOB bandwidth leads to improved robustness margin in general. The same conclusion is valid for the affine linear case where uncertain parameters are present both on the numerator and the denominator of the plant. The proposed method also allows analyzing the non-minimum phase case in which the DOB based systems may lose their superiorities.

The relation between bandwidth and robustness of disturbance observer is analytically derived using a spherical polynomial representation.

3. GUARDIAN MAP FOR ROBUST D-STABILITY

3.1 State of the Art

The basic idea of predicting the effects of the disturbances emerged in the late 60s [15], and the idea eventually led to the concept of extended state observer [16]. The disturbance observer structure as we use today is introduced by [17] to attenuate the effects of parameter changes, disturbances, and nonlinear effects. In addition, a different line of research called active disturbance rejection has shown that the problem of designing a controller can be viewed as a disturbance attenuation problem by putting the disturbances at the center of the design philosophy [21]. The reader may refer [70] for further discussion on similar design perspectives such as "unknown input observer", "extended state observer", and "uncertainty estimation". Although DOB has been successfully applied in industry, theoretical developments in design and analysis have lagged behind practical applications [31]. Especially in the context of the robust DOB design, H_∞ based conservative methods predominates the literature [43–45]. The first attempts to analyze DOB-based systems under uncertainty were made by [42] and [85]. In [72], an analysis of DOB-based control systems via spherical polynomials is presented. However, these studies do not cover the state-space domain and do not propose a state-space design method. It can be said that no special attention has been paid to the case where the perturbation is real and time-invariant in the state-space approach. Although approaches are made under the concept of quadratic stability for design under linear time-invariant perturbations, it is generally conservative in the matrix case [12]. To address this problem, the researchers turned their attention to a line of research called the guardian map approach, which suggested less conservative results. The guardian map approach is suggested by [86] and is utilized in application areas such as robotics [87] and flight control [88], as well as theoretical studies related to interval systems [89] and linear systems with real parameter uncertainty [90]. However, to the best of the author's knowledge, no attempt has been made to design a DOB-based system under real parameter uncertainty in state-space by using this approach. This study aims to fill this gap.

3.2 An Approach for Robust Stability

Consider

$$A(q) = A_0 + E(q) \tag{3.1}$$

Where A_0 is *n x n* Hurwitz matrix, and E is a constant uncertainty matrix which has the following form;

$$E = \sum_{i=1}^{p} q_i A_i \tag{3.2}$$

At this point Kronecker sum and Kronecker product have to be defined;

Kronecker Product: Let A be an n-dimensional matrix and B an m-dimensional matrix. The mn-dimensional matrix C defined as

$$A \otimes B = C = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \tag{3.3}$$

is called the Kronecker product of A and B.

Kronecker sum: The matrix $A \oplus B$ is called the Kronecker sum of A and B and is defined as follows;

$$A \oplus B = A \otimes I_m + I_n \otimes B \tag{3.4}$$

Theorem 3.1 [12] : *The system $A(q) = A_0 + E(q)$ is robustly stable if and only if*

$$\det\left(A_0 \oplus A_0 + \sum_{i=1}^{p} q_i (A_i \oplus A_i)\right) \neq 0 \tag{3.5}$$

Proof [90]: If the system is robustly stable, then

$$Re\ \lambda_i (A(q) \oplus A(q)) = Re\ \left(\lambda_j (A(q)) + \lambda_k (A(q))\right) < 0 \tag{3.6}$$

for every eigenvalue λ_i of $A(q) \oplus A(q)$.

To prove sufficiency, suppose the system is not robustly stable; for example, $A(q^*)$ has purely imaginary eigenvalues. Then,

$$\det\left(A\left(q^*\right) \oplus A\left(q^*\right)\right) = 0 \tag{3.7}$$

□

Note here that the problem is converted into a nonsingularity problem which can be handled by a symbolic algebra tool more easily. Before going into the analysis, a suitable state space approach should be stated first.

Let family $A = \{A(q) : q \in Q\}$ of nxn matrices with $A(q)$ depending continuously on q. Family A is robustly stable if and only if the family $\overline{A}$ is robustly nonsingular where;

$$\overline{A} \triangleq A \oplus A \tag{3.8}$$

Since the linear transformation on A preserves the affine linear dependence of matrix entities on uncertain parameters, the nonsingularity problem involves a polytope of matrices. However, linear transformation does not preserve the interval matrix structure. Therefore, the interval matrix family transforms into a polytope of matrices.

The following example from [77] shows this result;

$$A = \begin{bmatrix} q_{11} & 1 \\ 3 & q_{22} \end{bmatrix} \tag{3.9}$$

$$A \oplus A = \begin{bmatrix} 2q_{11} & 1 & 1 & 0 \\ 3 & q_{11}+q_{22} & 0 & 1 \\ 3 & 0 & q_{11}+q_{22} & 1 \\ 0 & 3 & 3 & 2q_{22} \end{bmatrix} \tag{3.10}$$

Before getting into detail on the uncertain systems, the definition of the guardian map has to be given.

Definition [86]: Let $v : R^{nxn} \to \mathbb{R}$ be a given mapping. Then v is said to guard on open set of nxn matrices $\mathbf{A}$ if $v(\mathbf{A}) \neq 0$ for $A \in \mathbf{A}$ and $v(\mathbf{A}) = 0$ for $A \in \delta\mathbf{A}$. For such cases, we call v a guardian map for $\mathbf{A}$.

For example, for $A \in R^{nxn}$,

$$v(A) = \det(A \oplus A) \tag{3.11}$$

is a guardian map for the set $\mathbf{A}$ of stable nxn matrices. The spectrum of A is $spec(A \oplus A) = \left\{\lambda_i + \lambda_j : \lambda_i, \lambda_j \in spec(A)\right\}$. So, the matrix $A \oplus A$ is nonsingular if, and only if, $\lambda_i + \lambda_j \neq 0,\ i, j = 1, 2, \ldots, n$

3.2.1 Uncertain matrices

Consider an uncertain matrix of the form;

$$\dot{x} = A(q)x, \qquad x(0) = x_0 \tag{3.12}$$

where

$$A(q) = \sum_{i=0}^{r} q_i A_i = A_0 + E(q) \tag{3.13}$$

The parameter vector $q^T = [q_1, q_2, \ldots, q_r]$ belongs the the hyper-rectangular set $\Omega(\beta)$ defined by:

$$\Omega(\beta) = (q \in R^r : q_i^0 - \beta w_i^{\min} \leq q_i \leq q_i^0 + \beta w_i^{\max}) \tag{3.14}$$

where $i \in \{1, 2, \ldots, r\}$, $\beta > 0$, and $w^{\min}$ *and* $w^{\max}$ for $i = 1, 2, \ldots, r$ are positive weights.

Depending on the characteristic of the $E(q)$, the family of matrices takes different names. Consider the following representation;

$$E(q) = \sum_{i=0}^{r} q_i A_i \tag{3.15}$$

where A_i are constant and with no restrictions on the structure. This type of representation produces a polytope of matrices in the matrix space. A special case of the polytope of matrices is independent variations case where;

$$E(q) = \sum_{i=0}^{r} q_i E_i \tag{3.16}$$

and E_i contains a single non-zero element at a different location in the matrix for each i. In this case, the set of possible $A(q)$ matrices forms a hyper-rectangle in matrix space [12]. In this representation, the family of matrices is labelled as an interval matrix family. In this study, interval matrix family representation is going to be used.

3.3 Guardian Map Based Disturbance Observer Design For D-Stability

After the introduction of the interval matrix family and guardian map approach, let us give the following corollary of Theorem 3.1;

Corollary 3.1 [12]**:** *The system*

$$\dot{x} = A(q)x, \qquad x(0) = x_0 \tag{3.17}$$

$$A(q) = \sum_{i=0}^{r} q_i A_i = A_0 + E(q) \tag{3.18}$$

is robustly stable if, and only if;

$$\det\left(A_0 \oplus A_0 + \sum_{i=0}^{r} q_i \left(A_i \oplus A_i\right)\right) \neq 0 \ \in \Omega \tag{3.19}$$

By utilizing the rationale behind the corollary, the following approach has been proposed by [91], where only the robust Hurwitz stability problem is considered, and the robust D-Stability is not addressed. Let us denote

$$\overline{A} \triangleq A \oplus A \tag{3.20}$$

$$\overline{A_q} \triangleq \sum_{i=0}^{r} q_i \left(A_i \oplus A_i\right) \tag{3.21}$$

Then, for robust Hurwitz stability, the following condition has to be satisfied;

$$\det\left(\overline{A} + \overline{A_q}\right) \neq 0 \tag{3.22}$$

Since A is assumed to be stable;

$$\det\left(\overline{A}\right) \neq 0 \tag{3.23}$$

Hence,

$$\det\left(\overline{A}\ \left(I + \overline{A}^{-1}\ \overline{A_q}\right)\right) \neq 0 \tag{3.24}$$

$$et\left(\overline{A}\right) \det\left(I + \overline{A}^{-1}\ \overline{A_q}\right) \neq 0 \tag{3.25}$$

$$\det\left(I + \overline{A}^{-1}\ \overline{A_q}\right) \neq 0 \tag{3.26}$$

If

$$\rho\left(\overline{A}^{-1}\overline{A_q}\right) < 1 \tag{3.27}$$

where ρ is the spectral radius, then $A(q)$ is robustly stable.

3.3.1 Path to a new guardian map

The previous method considers only robust Hurwitz stability. However, by adopting the rationale behind the approach, a new guardian map can be proposed for robust D-Stability.

Consider the following system to illustrate the proposed idea

$$\dot{x} = Ax + E(q) \tag{3.28}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{3.29}$$

$$E(q) = \begin{pmatrix} d_1 & 0 \\ 0 & d_4 \end{pmatrix} \tag{3.30}$$

$$E_r = r \begin{pmatrix} e^{iw} & 0 \\ 0 & e^{iw} \end{pmatrix} \tag{3.31}$$

It is possible to diagonalize the A matrix using the following transformation;

$$V = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \tag{3.32}$$

where v_1 *and* v_2 are the eigenvectors of A. Then, it is possible to write

$$D = V^{-1}AV \tag{3.33}$$

where D matrix is a diagonal matrix consisting of the eigenvalues.

If the diagonal case is used for simplicity following representation could be used for evaluating the D-stability;

$$A + E(q) - (D + E_r) \tag{3.34}$$

Using the Kronecker sum operation, the following definitions are made;

$$\overline{A} \triangleq A \oplus A \tag{3.35}$$

$$\overline{E}_q \triangleq E(q) \oplus E(q) \tag{3.36}$$

$$\overline{E_r} \triangleq E_r \oplus E_r \tag{3.37}$$

$$\overline{D} \triangleq D \oplus D \tag{3.38}$$

Finally, let us investigate the eigenvalues of the following matrix

$$\Lambda = \overline{A} + \overline{E_q} - \left(\overline{D} + \overline{E_r}\right) \tag{3.39}$$

The eigenvalues can be shown to be given as

$$\lambda_1 = d_1 + d_4 - 2e^{j\omega}r \tag{3.40}$$

$$\lambda_2 = d_1 + d_4 - 2e^{j\omega}r \tag{3.41}$$

$$\lambda_3 = 2\left(d_1 - e^{j\omega}r\right) \tag{3.42}$$

$$\lambda_4 = 2\left(d_4 - e^{j\omega}r\right) \tag{3.43}$$

where $i = \sqrt{-1}$. Especially λ_3 and λ_4 are particularly useful for evaluating the pole spread around the nominal poles. By adjusting the spread radius r, it can be calculated how much uncertainty can be tolerated for a given radius. If the uncertainty value d is equal to the given radius, some of the eigenvalues of Λ go to zero, and therefore the determinant in (3.19) goes to zero.

Since the problem is to cluster all eigenvalues inside a predefined region, it is convenient to represent the system in a diagonal form in which assigning the robustness-related function is easier.

To this end, consider the following representations;

$$\dot{x} = Ax + E(q) \tag{3.44}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{3.45}$$

$$E(q) = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \tag{3.46}$$

$$E_r = \begin{pmatrix} r_1 e^{j\omega} & 0 \\ 0 & r_2 e^{j\omega} \end{pmatrix} \tag{3.47}$$

It is possible to diagonalize the matrix $A + E(q)$ using the following transformation;

$$V = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \tag{3.48}$$

Where v_1 and v_2 are eigenvectors of A.

$$D_q = V^{-1}\left(A + E(q)\right)V \tag{3.49}$$

where D_q is a diagonal matrix consisting of eigenvalues of $A + E(q)$. Similar to the previous section, D matrix represents the diagonal form of the A matrix.

Now, let us transform the uncertain matrix $A + E(q)$ into the diagonal form and assign an uncertainty region to each eigenvalue.

$$V^{-1}\left(A + E(q)\right)V \; - (D + E_r) = D_q - D - E_r \tag{3.50}$$

$$= \begin{pmatrix} \lambda_1(q) & 0 \\ 0 & \lambda_2(q) \end{pmatrix} - \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - \begin{pmatrix} r_1 e^{j\omega} & 0 \\ 0 & r_2 e^{j\omega} \end{pmatrix} \tag{3.51}$$

Where $\lambda_{1,2}(q)$ represent eigenvalues of $A + E(q)$ and $\lambda_{1,2}$ represent eigenvalues of A. Note that the statement above is diagonal.

In order to convert the problem into a robust nonsingularity problem, the Kronecker sum operation is introduced.

$$\overline{D_q} = D_q \oplus D_q \tag{3.52}$$

$$\overline{D} = D \oplus D \tag{3.53}$$

$$\overline{E_r} = E_r \oplus E_r \tag{3.54}$$

Finally, let's investigate the eigenvalues of the following statement;

$$Det\left(\overline{D_q} - \overline{D} - \overline{E_r}\right) \tag{3.55}$$

If there is no uncertainty, $\overline{D_q} = \overline{D}$. Since $Det\left(\overline{E_r}\right) \neq 0$, then $Det\left(\overline{D_q} - \overline{D} - \overline{E_r}\right) \neq 0$. If the eigenvalues of $\overline{D_q}$ are equal to those of $\overline{D} + \overline{E_r}$ in the presence of uncertainty, then $Det\left(\overline{D_q} - \overline{D} - \overline{E_r}\right) = 0$. Actually, $\overline{D} + \overline{E_r}$ defines a circle around the eigenvalues of $\overline{D}$, in which the radius of the circle is determined by $\overline{E_r}$.

Although it is useful to define such a guardian map, it is also difficult to determine the singular points where the determinant vanishes, that is,

$$Det\left(\overline{D_q} - \overline{D} - \overline{E_r}\right) \neq 0,\ \forall E(q) \tag{3.56}$$

Since A is stable, so is D. Therefore,

$$\det(D \oplus D) \neq 0 \tag{3.57}$$

$$\det\left(\overline{D}\right) \neq 0 \tag{3.58}$$

Hence, $Det\left(\overline{D_q} - \overline{D} - \overline{E_r}\right) \neq 0$ is equivalent to,

$$Det\left(-I + \overline{D}^{-1}\left(\overline{D_q} - \overline{E_r}\right)\right) \neq 0 \tag{3.59}$$

This expression shows that if one of the eigenvalues of $\overline{D}^{-1}(\overline{D_q} - \overline{E_r})$ is 1, then the determinant vanishes, which means that one of the eigenvalues of the uncertain system crosses the predefined region border.

An interesting observation is that the distance between the eigenvalues of $\overline{D}^{-1}(\overline{D_q} - \overline{E_r})$ and 1 indicates how close an uncertain eigenvalue is to the predefined region border. This idea can be used for defining a cost function for D-stability.

One additional operation is required to isolate the more meaningful elements of $\overline{D}^{-1}(\overline{D_q} - \overline{E_r})$.

Spectrum of $\overline{X}$, where $\overline{X} = X \oplus X$ is $spec(X \oplus X) = \left\{\lambda_i + \lambda_j : \lambda_i, \lambda_j \in spec(X)\ \right\}$. Consequently, for 2x2 matrix X,

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.60}$$

$$spec(X \oplus X) = \{2\lambda_1,\ \lambda_1 + \lambda_2, \lambda_1 + \lambda_2, 2\lambda_2\} \tag{3.61}$$

Since it is more convenient to consider the first and the last eigenvalues, an isolation matrix can be defined.

Let M be an isolation matrix for 2x2 A matrix,

$$M_{2x2} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.62}$$

Then, $M\,\overline{D}^{-1}\left(\overline{D_q} - \overline{E_r}\right) M^T$ is a 2x2 matrix, and the eigenvalues of this matrix are related to the cost function for each eigenvalue of the uncertain system.

For a 3x3 A matrix, the isolation matrix is,

$$M_{3x3} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.63}$$

By following the same pattern, required isolation matrices can be found for higher-order systems.

3.3.2 The proposed guardian map

Although it is quite convenient to express the problem with the previous guardian map, it suffers from the direction search problem since the search algorithm has to investigate all directions of the term e^{iw} for the circular D-region case. This leads to an unnecessarily complex search algorithm. The new approach should rely directly on the distance without involving the direction. At this point, the Kronecker product may be more useful since the spectrum of the Kronecker product consists of the multiplication of all eigenvalues, that is,

$$spec(A \otimes B) = \lambda_i\, \mu_j, \quad i = 1, 2, \ldots n,\ j = 1, 2, \ldots m \tag{3.64}$$

where λ_i is an eigenvalue of A and μ_i is an eigenvalue of B. Multiplication of eigenvalues leads to the square of the distance. This idea may be useful for representing uncertainty radius.

Consider the following expression;

$$D_{KP} = \left(D_q - D\right) \otimes \left(D_q - D\right) \tag{3.65}$$

Here, $D_q$and D represent the uncertain diagonal matrix and the nominal diagonal matrix, respectively. Kronecker product of the difference matrix leads to the distance information between nominal and perturbed poles, that is,

$$\left(D_q - D\right) \otimes \left(D_q - D\right) = \left[\begin{pmatrix} \lambda_{q1} & 0 \\ 0 & \lambda_{q2} \end{pmatrix} - \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}\right] \otimes \left[\begin{pmatrix} \lambda_{q1} & 0 \\ 0 & \lambda_{q2} \end{pmatrix} - \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\right] \tag{3.66}$$

$$\left\{ \left(\lambda_{q1} - \lambda_1\right)^2, \left(\lambda_{q2} - \lambda_2\right)^2 \right\} \in spec\left\{D_{KP}\right\} \tag{3.67}$$

Now, in order to define an uncertainty region, the following matrix can be used;

$$E_r = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \tag{3.68}$$

Since the aim is to make the determinant zero when the uncertain poles cross the boundary, the following expression can be used;

$$\det\left[\left(E_r \otimes E_r\right)^{-1} \left(D_q - D\right) \otimes \left(D_q - D\right) - I\right] \neq 0 \tag{3.69}$$

Note that when there is no uncertainty, the expression is reduced to;

$$\det[-1] \neq 0 \tag{3.70}$$

Also, if the uncertain poles cross the boundary, one of the entities of

$$\left(E_r \otimes E_r\right)^{-1} \left(D_q - D\right) \otimes \left(D_q - D\right) \tag{3.71}$$

is reduced to 1, and therefore the determinant is zero.

Theorem 3.2: *Let $A_q = A_0 + E_q$ be a family of uncertain matrices, where A_0 is the $n \times n$ nominal matrix with no eigenvalues on the imaginary axis, and E_q is the perturbation matrix.Let,*

$$V = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \tag{3.72}$$

where v_1 and v_2 are eigenvectors of A_q.

$$D_q = V^{-1} A_q V \tag{3.73}$$

D is the diagonal matrix whose entities are eigenvalues of A_0.

$$E_r := \begin{pmatrix} r_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & r_n \end{pmatrix} \tag{3.74}$$

Then, A_q is D-stable if and only if

$$\det\left[(E_r \otimes E_r)^{-1}\left(D_q - D\right) \otimes \left(D_q - D\right) - I\right] \neq 0 \;\forall E_q \tag{3.75}$$

where; D regions are defined by circles with radii defined by E_r and their centers are defined by D.

Corollary 3.2 :

$$\det\left[(E_r \otimes E_r)^{-1}\left(D_q - D\right) \otimes \left(D_q - D\right) - I\right] \neq 0 \;\forall E_q \tag{3.76}$$

if

$$\rho\left[(E_r \otimes E_r)^{-1}\left(D_q - D\right) \otimes \left(D_q - D\right)\right] < 1 \tag{3.77}$$

where ρ is the spectral radius.

3.4 State Feedback Controller Design

The proposed approach can be applied to the state feedback problem. The following design guideline shall be followed;

1. Define $(A + E_q \;, B, C, D)$ matrices
2. Check controllability
3. Transform $A + E_q + B\;K$ into modal form to obtain D_q
4. Define the desired eigenvalue locations in diagonal form, D
5. Obtain following;

$$M\;(D_q - D) \otimes \left(D_q - D\right)(E_r \otimes E_r)^{-1} M^T \tag{3.78}$$

   where the matrix $M=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for 2×2 A matrix. Note that, depending on the size of A, the structure of M matrix is changed.

6. Define the uncertainty radius for each eigenvalue
7. Define the cost function J_p to minimize as follows;

$$J_p = Max\;\lambda\left(M\;\left(D_q - D\right) \otimes \left(D_q - D\right)(E_r \otimes E_r)^{-1} M^T\right)\;\forall E_q \tag{3.79}$$

8. Minimize J_p for K

3.4.1 Case study: State feedback controller design

Consider the following example,

$$\dot{x} = A_0 x + E(q) + Bu \tag{3.80}$$

$$u = Kx \tag{3.81}$$

$$A_0 = \begin{pmatrix} -5 & 3 \\ -3 & -1 \end{pmatrix} \tag{3.82}$$

$$E(q) = \begin{pmatrix} d_1 & 0 \\ 0 & d_4 \end{pmatrix} = \begin{pmatrix} [0,\ 1] & 0 \\ 0 & [0,\ 0.4] \end{pmatrix} \tag{3.83}$$

$$E_r = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \tag{3.84}$$

$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.85}$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{3.86}$$

Since Rank$\begin{pmatrix} B & AB \end{pmatrix} = 2$, the system is observable. Let,

$$D_q \triangleq Diag\left(A + E_q + B\ K\right) \tag{3.87}$$

such that

$$D_q = \begin{pmatrix} D_{q1} & 0 \\ 0 & D_{q2} \end{pmatrix} \tag{3.88}$$

where,

$$D_{q1} = \tfrac{1}{2}\left(-6 + d_1 + d_4 + k_1 + k_2 - \sqrt{a+b+c}\right) \tag{3.89}$$

$$a = d_1^2 + d_4^2 + 4\left(-5 + k_1 - k_2\right) \tag{3.90}$$

$$b = 2d_4\left(4 - k_1 + k_2\right) \tag{3.91}$$

$$c = -2d1\left(4 + d_4 - k_1 + k_2\right) + \left(k_1 + k_2\right)^2 \tag{3.92}$$

$$D_{q2} = \tfrac{1}{2}\left(-6 + d_1 + d_4 + k_1 + k_2 + \ \sqrt{d+e+f}\right) \tag{3.93}$$

$$d = d_1^2 + d_4^2 + 4\left(-5 + k_1 - k_2\right) \tag{3.94}$$

$$e = 2d_4\left(4 - k_1 + k_2\right) \tag{3.95}$$

$$f = -2d_1\left(4 + d_4 - k_1 + k_2\right) + \left(k_1 + k_2\right)^2 \tag{3.96}$$

The desired eigenvalue location matrix D is given below;

$$D = \begin{pmatrix} -10 & 0 \\ 0 & -2 \end{pmatrix} \tag{3.97}$$

Finally, the controller gains can be obtained as a result of the following optimization;

$$\begin{gathered} Min\ J_p = \\ Max\ \lambda\ \left(M\ \left(D_q - D\right) \otimes \left(D_q - D\right) \left(E_q \otimes E_q\right)^{-1} M^T\right) \forall E_q \\ subject\ to\ K \end{gathered} \tag{3.98}$$

After the optimization the following controller gains and objective function are obtained;

Table 3.1 : Controller gain and objective function.

Controller Gain	Objective Function
$\begin{pmatrix} 1.105 & -7.996 \end{pmatrix}$	0.806

Since the objective function given in Table 3.1, $J_p < 1$, all uncertain eigenvalues are contained in a predefined D-region.

The eigenvalue spread of the uncertain system is obtained as in Figure 3.1

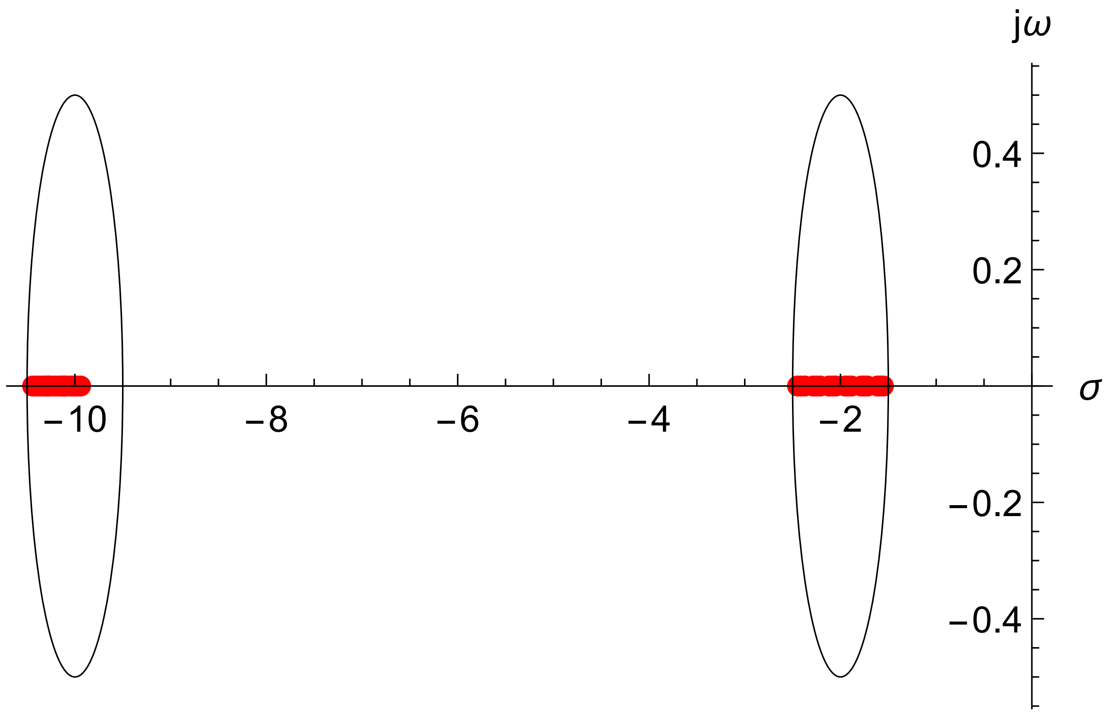

Figure 3.1 : Eigenvalue spread obtained for K^*.

3.5 State Space Representation of DOB

Finally, the proposed design method is applied to the disturbance observer design in this section. Assumptions: The following assumptions have been made [92];

1. The plant model is strictly proper
2. The plant model does not have zero at the origin
3. The disturbance model is known

Let the nominal system has the following form;

$$\dot{x} = Ax + B(u + d) \tag{3.99}$$

$$y = Cx \tag{3.100}$$

where $x \in \mathbb{R}^n$, and the system is both observable and controllable. The disturbance model is;

$$\dot{x_d} = A_d x_d \tag{3.101}$$

$$d = C_d x_d \tag{3.102}$$

Where $x_d \in \mathbb{R}^{n_d}$, (A_d, C_d) is observable, and A_d has at least one zero eigenvalue.

The observability condition can easily be understood in the context of a disturbance observer since the disturbance estimation is based on the system output signal.

At this point, it is also worth noting that the disturbance signal d and the control input u are assumed to affect the system via the same channels, which is called in the literature as the "matched condition". But in general, this may not be true. However, for the sake of simplicity more specific case has been considered at this point.

To express the complete system more compactly, the augmented plant, P_z, is given as follows;

$$\dot{x_a} = A_a\, x_a + B_a u \tag{3.103}$$

$$y_a = C_a\, x_a \tag{3.104}$$

where the augmented state vector is $x_a = \begin{bmatrix} x^T \ x_d^T \end{bmatrix}^T$, and the augmented system matrices are,

$$A_a = \begin{bmatrix} A & B\,C_d \\ 0 & A_d \end{bmatrix},\ B_a = \begin{bmatrix} B \\ 0 \end{bmatrix},\ C_a = \begin{bmatrix} C & 0 \end{bmatrix} \tag{3.105}$$

By assuming the augmented plant is observable;

$$\dot{\hat{x}}_a = A_a\,\hat{x}_a + B_a\,u + L(y_a - C_a\,\hat{x}_a) = (A_a + LC_a)\,x_a + \begin{bmatrix} B_a & L \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \tag{3.106}$$

$$\hat{d} = \begin{bmatrix} 0 & C_d \end{bmatrix} x_a \tag{3.107}$$

$$\begin{bmatrix} 0 & C_d \end{bmatrix} \triangleq C_{da} \tag{3.108}$$

where C_{da} is the augmented output matrix. The DOB structure in state space is given in Figure 3.2.

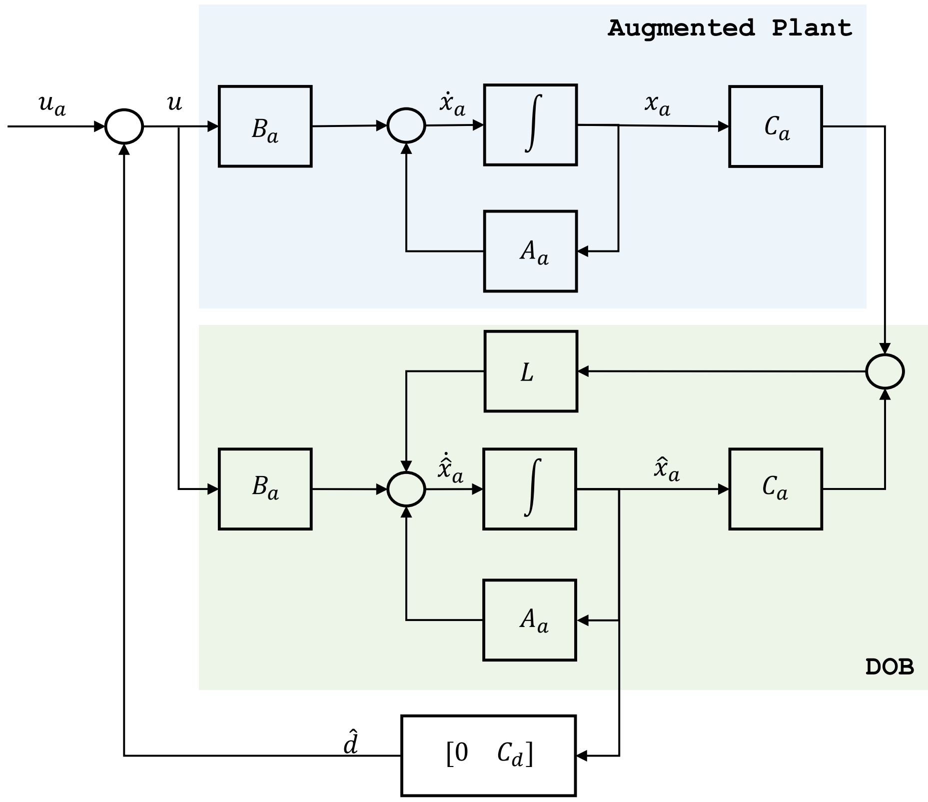

Figure 3.2 : Disturbance observer structure in state space.

3.5.1 Case study: Design of the DOB based on a new guardian map

Consider the following example for the system given in (3.103);

$$A_0 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.109}$$

$$E(q) = \begin{pmatrix} d_1 & 0 \\ 0 & d_4 \end{pmatrix} = \begin{pmatrix} [0,\ 0.5] & 0 \\ 0 & [-0.5,\ 0.5\,] \end{pmatrix} \tag{3.110}$$

$$E_r = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} = \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{3.111}$$

$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.112}$$

$$C = \begin{pmatrix} 1 & 1 \end{pmatrix} \tag{3.113}$$

$$A_d = (0) \tag{3.114}$$

$$C_d = (1) \tag{3.115}$$

$$L = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \tag{3.116}$$

Then,

$$A + LC = \begin{pmatrix} -2 + d_1 + l_1 & l_1 & 1 \\ l_2 & -1 + d_4 + l_2 & 1 \\ l_3 & l_3 & 0 \end{pmatrix} \tag{3.117}$$

The desired eigenvalue locations are assumed to be -20, -3, and -3. Following the design steps outlined in Section 3.4, the observer gain matrix is found to be;

$$L = \begin{pmatrix} -25.6938 \\ 2.79406 \\ -42.6519 \end{pmatrix} \tag{3.118}$$

The same D-region (a disk centered at -3 with radius 1.5) is assigned to the dominant eigenvalues. However, the D-region for the third eigenvalue (a disk centered at -20 with radius 2) is left more relaxed. Figure 3.3 shows the eigenvalue spread of the design. It is one of the main advantages of the proposed design method that the uncertainty regions can be assigned to specific nominal eigenvalues. In the DOB design example, more strict D-regions are assigned to the dominant eigenvalues, both of which are at -3. Hence, the objective of the design is to restrict dominant eigenvalues to these regions.

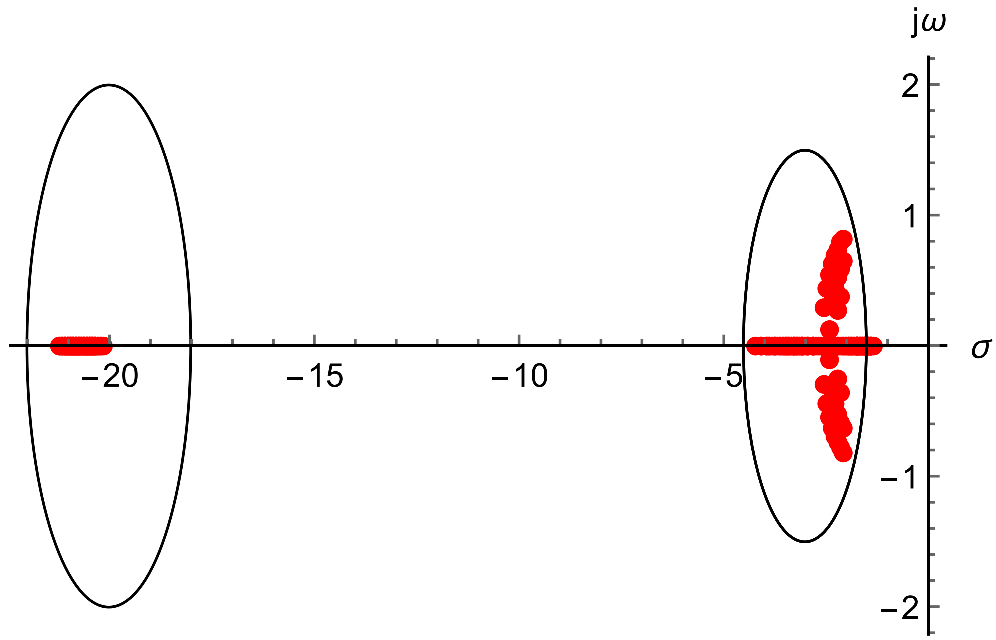

Figure 3.3 : Eigenvalue spread of the proposed design.

3.5.2 Conclusion

This study proposes a new way of designing DOB for uncertain systems by a novel guardian map. The main contribution of the study is that it allows constraining uncertain eigenvalues into separate predefined D-regions. The predefined regions for nominal eigenvalues can be different from each other. As a result, different robustness criteria are assigned for each of the nominal eigenvalues. However, the curse of dimensionality is the main drawback of the method, and handling it is considered in future work.

4. DISTURBANCE OBSERVER DESIGN BY EXTENDED LOOSE EIGENSTRUCTURE ASSIGNMENT FOR DISJOINT D-REGION STABILITY

4.1 Introduction

Previous chapter show that the proposed guardian map is useful for representing the problem. However, the method is computationally ineffective and this situation gets even worse when the system order is high. Therefore, we aim to represent the problem in a different framework. The aim of this study is to cluster the closed loop eigenvalues in a prescribed design region such a way that design regions are defined around the desired closed loop eigenvalues. Therefore, the design regions should be disjoint as we have studied in the guardian map approach. Generally, there is a useful LMI representation for D-stabilization in which the various LMI regions can be defined. Let us review the classical results of general D-stabilization in the LMI framework based on the work of [65].

Let $D = D_{L,M}$ be an LMI region, whose characteristic function is

$$F_D = L + sM + \bar{s}M^T \tag{4.1}$$

Design a state feedback control law $u = Kx$ for the linear system

$$\dot{x} = Ax + Bu \tag{4.2}$$

Such that closed loop system is $D_{L,M}$ stable.

The matrix $A + BK$ is $D_{L,M}$ stable if and only if there exist a symmetric positive definite matrix P such that;

$$R_D(A,P) = L \otimes P + M \otimes ((A+BK)P) + M^T \otimes ((A+BK)P)^T < 0 \tag{4.3}$$

Where $\otimes$ represents Kronecker product. Let $W = KP$,

$$L \otimes P + M \otimes AP + M^T \otimes PA^T + M \otimes BW + M^T \otimes W^T B^T < 0 \tag{4.4}$$

Then the control matrix K is defined as follows;

$$K = WP^{-1} \tag{4.5}$$

The most common LMI regions are given in Figure 4.1. This method clusters all closed loop eigenvalues in a connected region and does not consider the eigenstructure assignment. When the assignment regions are not common and disjoint, assignment condition cannot be reduced to a well-known Lyapunov inequality. This is the main difficulty of working with the disjoint regions. The aim is to cluster uncertain eigenvalues into a disjoint region by using eigenstructure assignment. In the eigenstructure assignment, eigenvalues are assigned strictly to desired points. However, instead of assigning eigenvalues to specific point, designer may utilize design regions so that there is much more design freedom left for the further control objectives.

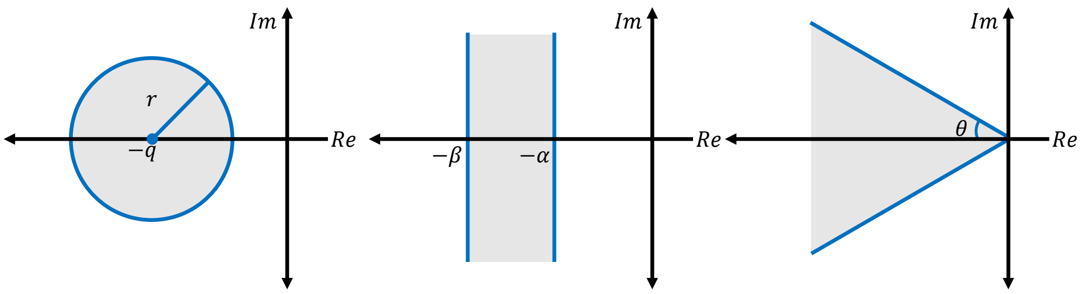

Figure 4.1 : LMI regions.

4.2 Preliminaries

4.2.1 Eigenstructure assignement approach

The above discussion motivates to introduce a new regional assignment approach to the eigenstructure assignment problem. In this study, aim is to constrain each closed loop eigenvalue to individual regions. Consider the LTI system:

$$\dot{x} = Ax + Bu \tag{4.6}$$

$$u = Kx \tag{4.7}$$

where $A \in \mathbb{C}^{nxn}$, $B \in \mathbb{C}^{nxm}$. In the classical eigenstructure assignment, desired closed loop eigenvalues, λ_i, are fixed in the beginning of the design. If there exist gain parameters $\xi_i, \zeta_i \in \mathbb{C}^n x \mathbb{C}^m,\ \xi_i \neq 0$, such that,

$$\begin{bmatrix} \lambda_i I - A & B \end{bmatrix} \begin{bmatrix} \xi_i \\ \zeta_i \end{bmatrix} = 0 \tag{4.8}$$

holds on $R \triangleq [\xi_i \ldots \xi_n]$ is nonsingular, a desired state feedback is determined as

$$K = QR^{-1} \tag{4.9}$$

where $Q = [\zeta_i \ldots \zeta_n]$. Here, ξs are closed loop eigenvectors. At this point, in order to obtain more design freedom, methods which is proposed in the literature divided into two different lines of research;

1. Strict partial eigenvalue assignment [62], [93]

2. Regional assignment [63]

4.2.2 Previously proposed method

A quadratic region in a complex plane is defined as follows;
For a given $\Theta = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \gamma \end{bmatrix} \in H_2$, $\det \Theta < 0$, a closed region is defined as follows [66];

$$D_\Theta = \left\{ \lambda \in C | \ \begin{bmatrix} 1 & \lambda \end{bmatrix} \Theta \begin{bmatrix} 1 \\ \overline{\lambda} \end{bmatrix} \geq 0 \right\} \tag{4.10}$$

For example, a closed disc with center: $\lambda_c \in C$, radius: $r > 0$ is defined as follows;

$$\alpha = r^2 - |\lambda_c|^2, \beta = \lambda_c, \gamma = -1 \tag{4.11}$$

If $\gamma \leq 0$, then D_Θ is convex, and if Θ is real symmetric, then D_Θ is symmetric with respect to the real axis. Finally, a region D_Θ which is convex and symmetric wrt real axis can be represented by an LMI region.
Lemma-1 in [66] states that, two column vectors $p, q \in C^n$ with q non-zero satisfy;

$$\begin{bmatrix} q & p \end{bmatrix} \begin{bmatrix} a & b^* \\ b & c \end{bmatrix} \begin{bmatrix} q & p \end{bmatrix}^* \geq 0 \tag{4.12}$$

if and only if $p = sq$ for some $s \in D^C$.
The result of this lemma leads to an LMI relation to determine if there exists an eigenvalue of matrix A in the region defined by a, b and c. Consider

$$(\lambda I - A)u \ = \ Bv \tag{4.13}$$

$$\lambda I u \ = \ Au + Bv \tag{4.14}$$

Let $q \triangleq u$ and $p \triangleq Au+Bv$, then;

$$\begin{bmatrix} u & Au+Bv \end{bmatrix} (\Theta) \begin{bmatrix} u^* \\ (Au+Bv)^* \end{bmatrix} \geq 0 \tag{4.15}$$

Then, for a given D_Θ, $A \in C^{nxn}$, the following statements are equivalent [66];

- $\exists \lambda \in D_\Theta$ s.t. $(\lambda I - A)u = 0$ for some $u \in C^n$, $u \neq 0$
- $\exists X \in H_n$ s.t.

$$X \neq 0, X \geq 0 \tag{4.16}$$

$$\begin{bmatrix} I & A \end{bmatrix} (\Theta \otimes X) \begin{bmatrix} I \\ A^* \end{bmatrix} \geq 0 \tag{4.17}$$

Finding an X such that the above inequalities are satisfied is an LMI problem. By solving this feasibility problem, it is possible to check whether A has at least one eigenvalue in D_Θ or not. The following theorem gives an elegant method which utilizes this property;

Theorem 4.1. [67]: *Consider the system;*

$$\dot{x} = Ax + Bu \tag{4.18}$$

$$u = Kx \tag{4.19}$$

where $A \in \mathbb{C}^{nxn}$, $B \in \mathbb{C}^{nxm}$. Let D_{Θ_i}, $i = 1, \ldots, n$ is given in the form of

$$\Theta_i = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \gamma \end{bmatrix} \in H_2 \tag{4.20}$$

Assume B has full column rank and $\gamma < 0$. There exist pairs (u_i, v_i) satisfies flowing;

$$\begin{bmatrix} \lambda_i I - A & -B \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} = 0 \tag{4.21}$$

for some $\lambda_i \in D_{\Theta_i}$, if and only if there exists $Z_i \in H_{n+m}$ such that;

$$\begin{bmatrix} I & 0 & A & B \end{bmatrix} (\Theta_i \otimes Z_i) \begin{bmatrix} I \\ 0 \\ A^* \\ B^* \end{bmatrix} \geq 0 \tag{4.22}$$

$$Z \neq 0 \tag{4.23}$$

$$Z \geq 0 \tag{4.24}$$

In [67], the equivalence between (4.22) and the existence of a Z_i, such that rank $Z_i = 1$ is shown. The aim of this study is to show similar results can be obtained without solving the rank-constrained optimization by introducing auxiliary LMI conditions.

A method to find controller gains is also proposed by [67]. Once the $Z \geq 0$ is found, a full rank factorization of Z is made as follows;

$$Z = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U^* & V^* \end{bmatrix} \tag{4.25}$$

In this case, (4.22) can be rewritten as follows;

$$\left\{(AU+BV)+\frac{\beta}{\gamma}U\right\}\left\{(AU+BV)+\frac{\beta}{\gamma}U\right\}^* - c^2UU^* \leq 0 \tag{4.26}$$

where $c \triangleq -\frac{\det\Theta}{\gamma^2}$, and $\Theta = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \gamma \end{bmatrix} \in H_2$.

Furthermore, by following Lemma-3 in [94], it is possible to find a matrix W, that is,

$$(AU+BV)+\frac{\beta}{\gamma}U = cUW \tag{4.27}$$

where, the right eigenvectors ξ of W lead to finding closed loop system eigenvector, that is, $u \triangleq U\xi$ and $v \triangleq V\xi$. Lemma-3 in [94] is given in Appendix A1.

Taking arbitrary right eigenvector $\xi \neq 0$ and eigenvalue λ of W,

$$cUW\xi\xi^*W^*U^*c = \left\{(AU+BV)+\frac{\beta}{\gamma}U\right\}\xi\xi^*\left\{(AU+BV)+\frac{\beta}{\gamma}U\right\}^* \tag{4.28}$$

Since $|\lambda|^2 < 1$, then,

$$cUW\xi\xi^*W^*U^*c = c^2\lambda^2U\xi\xi^*U^* \leq c^2U\xi\xi^*U^* \tag{4.29}$$

Finally,

$$\left\{(AU+BV)+\frac{\beta}{\gamma}U\right\}\xi\xi^*\left\{(AU+BV)+\frac{\beta}{\gamma}U\right\}^* \leq c^2U\xi\xi^*U^* \tag{4.30}$$

By taking $u \triangleq U\xi$ and $v \triangleq V\xi$, the initial inequality is obtained.

$$\begin{bmatrix} u & Au+Bv \end{bmatrix} \theta \begin{bmatrix} u^* \\ Au+Bv \end{bmatrix} \geq 0 \tag{4.31}$$

Since $R \triangleq [u_1 \dots u_n]$ is nonsingular, the desired state feedback is determined as $K = QR^{-1}$ w here $Q = [v_1 \dots v_n]$. Here, u_is are closed-loop eigenvectors.

This elegant method, called loose eigenstructure assignment method, which is proposed by [67] is successful in the case of nominal eigenstructure assignment. However, in the presence of uncertainties, it is not possible to find an W matrix by using Lemma-1 in [94]. Therefore right eigenvectors of W cannot be used as a candidate eigenvector to solve the eigenstructure assignment problem. This study introduces a novel method for designing DOB in the presence of parametric uncertainties. The method does not guarantee the existence of the common eigenvector that restrict all uncertain eigenvalues into the predetermined design regions, but it proposes a systematic approach to find a candidate eigenvectors.

4.3 Problem Formulation

The main purpose of the study is to extend the result of loose eigenstructure assignment to the uncertain matrix case. The following corollary of Theorem 4.1 is introduced for further investigation.

Corollary 4.1: *Consider the system;*

$$\dot{x} = A(\delta)x + Bu \tag{4.32}$$

$$u = Kx \tag{4.33}$$

where $A(\delta) \in \mathbb{C}^{nxn}$, $B \in \mathbb{C}^{nxm}$ *and*

$$A(\delta) \triangleq A_0 + \Delta A(\delta) \tag{4.34}$$

$$\Delta A(\delta) \triangleq \delta_1 A_1 + \delta_2 A_2 + \ldots + \delta_k A_k,\ \delta = \begin{Bmatrix} \delta_1 & \delta_2 & \ldots & \delta_k \end{Bmatrix} \in \Delta \tag{4.35}$$

$$\Delta \triangleq \delta | \textstyle\sum_{i=1}^{k} \delta_i = 1,\ \delta_i(t) \geq 0,\ i = 1, 2, \ldots, k \tag{4.36}$$

where A_0 *is the nominal matrix and* $A_i \in \mathbb{R}^{nxn}$, $i = 1, 2, \ldots k$ *represents the perturbation directions.*

Let D_{Θ_i}, $i = 1, \ldots, n$ *is given in the form of*

$$\Theta_i = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \gamma \end{bmatrix} \in H_2 \tag{4.37}$$

then, there exist pairs (u_i, v_i) *satisfy flowing;*

$$\begin{bmatrix} \lambda_i I - (A_0 + A_j) & -B \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} = 0,\ \ j = 0, 1, \ldots, k \tag{4.38}$$

for some $\lambda_i \in D_{\Theta_i}$, if there exists a common $Z_i \in H_{n+m}$ such that;

$$\begin{bmatrix} I & 0 & A_0+A_1 & B \end{bmatrix} (\Theta_i \otimes Z) \begin{bmatrix} I \\ 0 \\ (A_0+A_1)^* \\ (B)^* \end{bmatrix} \geq 0 \tag{4.39}$$

$$\vdots$$

$$\begin{bmatrix} I & 0 & A_0+A_k & B \end{bmatrix} (\Theta_i \otimes Z) \begin{bmatrix} I \\ 0 \\ (A_0+A_k)^* \\ B^* \end{bmatrix} \geq 0,\ Z \neq 0,\ Z \geq 0 \tag{4.40}$$

Since the intersection of convex sets is also convex, it is possible to search for a common positive semi-definite matrix Z for all matrix family.

Proof: If there exist a common $Z \neq 0,\ Z > 0$, then the inequalities of

$$\begin{bmatrix} I & 0 & A_0+A_j & B \end{bmatrix} (\Theta_i \otimes Z) \begin{bmatrix} I \\ 0 \\ \left(A_0+A_j\right)^* \\ B^* \end{bmatrix} \geq 0, \tag{4.41}$$

are satisfied for all $j = 1,2,\ldots,k$. By Theorem 4.1, the following equality is satisfied,

$$\begin{bmatrix} \lambda_i I - (A_0+A_j) & -B \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} = 0, \quad j = 1,2,\ldots,k \tag{4.42}$$

for some $\lambda_i \in D_{\Theta_i}$.

□

4.4 Main Results

Corollary 4.1 is the first step of the proposed approach. Consider the system in (4.32) and (4.33). If Corollary 4.1 is satisfied, then $\exists Z \geq 0$ and Z can be rewritten using a full rank decomposition as follows;

$$Z = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U^* & V^* \end{bmatrix} \tag{4.43}$$

$$\left\{((A_0+A_1)U+BV)+\frac{\beta}{\gamma}U\right\}\left\{((A_0+A_1)U+BV)+\frac{\beta}{\gamma}U\right\}^* - c^2UU^* \leq 0 \tag{4.44}$$

$$\vdots$$

$$\left\{((A_0+A_k)U+BV)+\frac{\beta}{\gamma}U\right\}\left\{((A_0+A_k)U+BV)+\frac{\beta}{\gamma}U\right\}^*-c^2UU^*\leq 0 \quad (4.45)$$

Let us use the following definitions;

$$F_i \triangleq ((A_0+A_i)U+BV)+\frac{\beta}{\gamma}U \quad (4.46)$$

$$G \triangleq \text{cU} \quad (4.47)$$

As stated in the previous section, it is not usually possible to find a common W [94], such that $WW^* \leq I$ and $F_i = GW$ for all $i = 1,2,\ldots k$. To find a common eigenvector for the matrix family, the following step is proposed.

Theorem 4.2. *Let $P \in H_{n+m}$ and the singular value decomposition of P be given as follows;*

$$P = U_pS_pV_p^* \quad (4.48)$$

where

$$U_p = \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_{n+m} \end{bmatrix} \quad (4.49)$$

If there exists $P > 0$ satisfying following inequalities;

$$FPF^* \leq GPG^* \quad (4.50)$$

then there exists at least one f and g pair that satisfies $||f|| \leq ||g||$, where f and g are a rank one matrices in the form of

$$f \triangleq \sigma_iF\xi_i\xi_i^*F \quad (4.51)$$

$$g \triangleq \sigma_iG\xi_i\xi_i^*G \quad (4.52)$$

and ξ_i is the i^{th} column of the U_p.

Proof: If there exists $P > 0$ satisfying the following inequality;

$$FPF^* \leq GPG^* \quad (4.53)$$

then the singular value decomposition of P can be written as follows;

$$P = U_pS_pV_p^* = U_pS_pU_p^* \quad (4.54)$$

Since P is positive definite $U_p = V_p$. Let us write rank one decomposition of P as follows;

$$U_p S_p U_p^* = \sigma_1 \xi_1 \xi_1^* + \sigma_2 \xi_2 \xi_2^* + \ldots + \sigma_{n+m} \xi_{n+m} \xi_{n+m}^* \tag{4.55}$$

then,

$$\begin{array}{c} F\left(\sigma_1 \xi_1 \xi_1^* + \ldots + \sigma_{n+m} \xi_{n+m} \xi_{n+m}^*\right) F^* \\ -G(\sigma_1 \xi_1 \xi_1^* + \ldots + \sigma_{n+m} \xi_{n+m} \xi_{n+m}^*) G^* \leq 0 \end{array} \tag{4.56}$$

$$\begin{array}{c} \sigma_1 \left\{F\left(\xi_1 \xi_1^*\right) F^* - G\left(\xi_1 \xi_1^*\right) G^*\right\} + \\ \ldots + \sigma_{n+m} \left\{F\left(\xi_{n+m} \xi_{n+m}^*\right) F^* - G\left(\xi_{n+m} \xi_{n+m}^*\right) G^*\right\} \leq 0 \end{array} \tag{4.57}$$

Since $\xi_i \xi_i^*$ is a rank one matrix, $F\left(\xi_i \xi_i^*\right) F^*$ and $G\left(\xi_i \xi_i^*\right) G^*$ matrices are also rank one which means that there is only one nonzero eigenvalue for those matrices. Following definitions are necessary for further investigation;

$$f_i \triangleq \sigma_i F \xi_i \xi_i^* F \tag{4.58}$$

$$g_i \triangleq \sigma_i G \xi_i \xi_i^* G \tag{4.59}$$

By introducing reasonable conservatism, frobenius norm of each component in (4.57) can be taken as follows;

$$(||f_1|| - ||g_1||) + (||f_2|| - ||g_2||) + \ldots (||f_{n+m}|| - ||g_{n+m}||) \leq 0 \tag{4.60}$$

Since the overall inequality is satisfied already, then at least one component of the inequality, that is,

$$(||f_k|| - ||g_k||)\,,\ k = 1, 2, .., n+m \tag{4.61}$$

has to be less or equal to zero.

□

Corollary 4.2: *If there exists a common ξ_i that satisfies $||f_i|| \leq ||g_i||$ for all inequalities given below;*

$$F_j\left(\sigma_1 \xi_1 \xi_1^* + \ldots + \sigma_{n+m} \xi_{n+m} \xi_{n+m}^*\right) F_j^* - G\left(\sigma_1 \xi_1 \xi_1^* + \ldots + \sigma_{n+m} \xi_{n+m} \xi_{n+m}^*\right) G^* \leq 0 \tag{4.62}$$

where $j = 1, 2, \ldots, k$, then u_i is the best candidate eigenvector derived from the positive definite matrix P in order to satisfy the following statement for all $A_0 + A_j$, $j = 1, 2, \ldots, k$,

$$\begin{bmatrix} \lambda_i I - (A_0 + A_j) & -B \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0 \tag{4.63}$$

where $\lambda_i \in \mathbf{D}$, $u \triangleq U\xi$, $v \triangleq V\xi$.

Note that this method does not guarantee $\sigma_i F\xi_i\xi_i^* F \leq \sigma_1 G\xi_i\xi_i^* G$. However, the candidate eigenvector derived from the singular value decomposition of P is the closest eigenvector which makes the inequality $\sigma_i F\xi_i\xi_i^* F \leq \sigma_1 G\xi_i\xi_i^* G$ valid.

Theorem 4.2 and Corollary 4.2 give a design method for robust eigenstructure assignment.

4.4.1 Design steps

1. Check Corollary 4.1 to determine if there exist a $Z \neq 0,\ Z \geq 0,\ Z \in H_{n+m}$

2. Using full rank decomposition of Z, that is,

$$Z = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U^* & V^* \end{bmatrix} \tag{4.64}$$

   define the following expressions;

$$F_i \triangleq \left\{ (A_i U + BV) + \frac{\beta}{\gamma} U \right\} \tag{4.65}$$

$$G \triangleq cU \tag{4.66}$$

3. Check Theorem 4.2 for $P > 0$. If there exists $P > 0$ obtain singular value decomposition of P, that is,

$$P = U_p S_p U_p^* \tag{4.67}$$

4. If there exists a common ξ (a column of U_p) satisfying the conditions of Corollary 4.2, then calculate the controller gain as follows;

$$u \triangleq U\xi \text{ and } v \triangleq V\xi \tag{4.68}$$

$$R \triangleq [u_1 \ldots u_n]\ ,\ Q = [v_1 \ldots v_n] \tag{4.69}$$

$$K = QR^{-1} \tag{4.70}$$

Note that, a unique design region Θ can be defined for each nominal eigenvalue. This allows defining more strict conditions for the dominant eigenvalues so that, eigenvalues that are not close to the dominant region can be left to lie in a more relaxed design region. This feature is especially useful when high-order systems are of concern.

Design for nominal systems has already been proposed by [67]. With this novel method, we extended the related theory by showing that loose eigenstructure assignment is also applicable to uncertain systems.

The method in the literature requires one LMI solution. Furthermore, If the W matrix is found, all eigenvectors of W can be used for the design. However, such a design method is not suitable for the simultaneous stabilization of matrix families. On the other hand, the proposed method in this paper employs a two-stage design in which simple LMI solutions are required throughout the design process. Furthermore, there is no guarantee that more than one eigenvector can be found, but at least one eigenvector of the P matrix satisfies the condition. The most important feature of the proposed method is that it is suitable design method in the presence of parametric uncertainties.

The proposed method has no advantage over the method in [67] for the design of the nominal system. However, the proposed method can also be applied to the uncertain system with parametric uncertainty while, the method in [67] cannot.

4.4.2 Design of disturbance observer in the presence of the parametric uncertainty

Assumptions: Following assumptions have been made [92];

- The plant model is strictly proper
- The plant model does not have zero at the origin
- The disturbance model is known

Let P_n has the following form;

$$\dot{x} = Ax + B(u+d) \tag{4.71}$$

$$y = Cx \tag{4.72}$$

where $x \in \mathbb{R}^n$, and the system is both observable and controllable. The disturbance model is;

$$\dot{x_d} = A_d x_d \tag{4.73}$$

$$d = C_d x_d \tag{4.74}$$

where $x_d \in \mathbb{R}^{n_d}$, (A_d, C_d) is observable, and A_d has at least one zero eigenvalue. The state-space representation of the DOB is already given in Figure 3.2.

The observability condition can easily be understood in the context of disturbance observer since the disturbance estimation is based on the system output signal.

At this point, it is also worth noting that disturbance signal d and the control input u are assumed to affect the system via the same channels, which is called in the literature a "matched condition". But in general, this may not be true. However, for the sake of simplicity, more specific case has been considered at this point.

To express the complete system more compactly, augmented plant, P_z, is given as follows;

$$\dot{x_a} = A_a x_a + B_a u \tag{4.75}$$

$$y_a = C_a x_a \tag{4.76}$$

where augmented state vector is $x_a = \begin{bmatrix} x^T & x_d^T \end{bmatrix}^T$, and augmented system matrices are,

$$A_a = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix},\ B_a = \begin{bmatrix} B \\ 0 \end{bmatrix},\ C_a = \begin{bmatrix} C & 0 \end{bmatrix} \tag{4.77}$$

By assuming the augmented plant is observable;

$$\dot{\widehat{x}}_a = A_a\,\widehat{x}_a + B_a\,u + L\,(y_a - C_a\,\widehat{x}_a) = (A_a - LC_a)\,x_a + \begin{bmatrix} B_a & L \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \tag{4.78}$$

$$\widehat{d} = \begin{bmatrix} 0 & C_d \end{bmatrix} \widehat{x}_a \tag{4.79}$$

$$\begin{bmatrix} 0 & C_d \end{bmatrix} \triangleq C_{da} \tag{4.80}$$

where C_{da} is the augmented output matrix.

The relation between time domain and frequency domain representations of the disturbance observer is given as follows [92];

$$\widehat{d} = -G_1(s)\,u + G_2(s)\,y \tag{4.81}$$

where

$$G_1(s) = \left[\begin{array}{c|c} A_a + LC_a & -B \\ \hline C_{\mathrm{da}} & 0 \end{array}\right],\ G_2(s) = \left[\begin{array}{c|c} A_a + LC_a & L \\ \hline C_{\mathrm{da}} & 0 \end{array}\right] \tag{4.82}$$

Considering the frequency domain disturbance observer equations;

$$\widehat{d} = -Q(s)\,u + Q(s)\,P_n^{-1}(s)\,y \tag{4.83}$$

Error dynamics can be written as follows;

$$\dot{e} = (A(\delta) - LC)\,e - (A - A(\delta))\,\widehat{x}_a \tag{4.84}$$

Obviously, if $\lambda\{A(\delta) - LC\} \in D$, then the system with disturbance observer performance is determined by the eigenvalue spread which is constrained by the design region D. The DOB design in the presence of the parametric uncertainty is based on the design steps given in section 4.4.1 in which augmented system in (4.77) is utilized.

4.4.3 Disturbance observer design for reduced-order case

The motivation for studying reduced order case is twofold; the first reason is obvious, there are some cases where all states are not observable [95]. Secondly, even if all states are observable, the designer may want to utilize low order DOB which is more desirable for number of reasons such as fast disturbance estimation and low complexity. For discrete system, a DOB design has been proposed in [96].

Consider the system

$$\dot{x} = Ax + B(u + d) \tag{4.85}$$

$$y = Cx \tag{4.86}$$

where $x \in R^n$ and the disturbance model is;

$$\dot{x_d} = A_d x_d \tag{4.87}$$

$$d = C_d x_d \tag{4.88}$$

where $x_d \in \mathbb{R}^{n_d}$, (A_d, C_d) is observable, and A_d has at least one zero eigenvalue.

As it is given in (4.75), the augmented plant is given as follows;

$$\dot{x_a} = A_a x_a + B_a u \tag{4.89}$$

$$y_a = C_a x_a \tag{4.90}$$

where augmented state vector is $x_a = \begin{bmatrix} x^T & x_d^T \end{bmatrix}^T$, and augmented system matrices are,

$$A_a = \begin{bmatrix} A & BC_d \\ 0 & A_d \end{bmatrix},\; B_a = \begin{bmatrix} B \\ 0 \end{bmatrix},\; C_a = \begin{bmatrix} C & 0 \end{bmatrix} \tag{4.91}$$

In this case, if the observability matrix of (A_a,C_a) has rank$\leq n+n_d$, where $n+n_d$ is the size of A_a, then there exists a similarity transformation such that;

$$\begin{aligned}\overline{A}_a &= TA_aT^T &(4.92)\\ \overline{B}_a &= TB_a &(4.93)\\ \overline{C}_a &= CT^T &(4.94)\end{aligned}$$

where T is unitary and the transformed system has unobservable modes, if any, in the upper left corner. The transformed matrices are given below;

$$\begin{aligned}\overline{A}_a &= \begin{bmatrix} A_{a_{no}} & A_{12} \\ 0 & A_{a_o} \end{bmatrix} &(4.95)\\ \overline{B}_a &= \begin{bmatrix} B_{a_{no}} \\ B_{a_o} \end{bmatrix} &(4.96)\\ \overline{C}_a &= \begin{bmatrix} 0 & C_{a_o} \end{bmatrix} &(4.97)\end{aligned}$$

where (C_{a_o},A_{a_o}) is observable, and the eigenvalues of $A_{a_{no}}$ are the unobservable modes. Now, only observable part is considered during the design process. Consider the transformed system given below;

$$\begin{aligned}\dot{z} &= \overline{A}_a(\delta)z+\overline{B}_au &(4.98)\\ y &= \overline{C}_az &(4.99)\end{aligned}$$

where $\overline{A}_a \in \mathbb{C}^{(n+n_d)x(n+n_d)}$, $\overline{B}_a \in \mathbb{C}^{(n+n_d)x(m)}$.Let,

$$\overline{A}_a(\delta) = \overline{A}_{ao}+\overline{A}_1+\cdots+\overline{A}_k \qquad (4.100)$$

where

$$\begin{aligned}\overline{A}_1 &= TA_1T^T &(4.101)\\ \overline{A}_k &= TA_kT^T &(4.102)\end{aligned}$$

Let $D_{\Theta i}$, $i=1,\ldots,n$ is given in the form of

$$\Theta_i = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \gamma \end{bmatrix} \in H_2 \qquad (4.103)$$

There exists pairs (u_i,v_i) satisfies following;

$$\begin{bmatrix} \lambda_i I-(\overline{A}_{ao}+\overline{A}_i)^* & -\overline{C}_a^T \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} = 0 \qquad (4.104)$$

For some $\lambda_i \in D_{\Theta_i}$, if there exists a common $Z_i \in H_{n+n_d+m}$ such that;

$$\begin{bmatrix} I & 0 & (\overline{A}_{ao}+\overline{A}_1)^* & \overline{C}_a^T \end{bmatrix} (\Theta_i \otimes Z) \begin{bmatrix} I \\ 0 \\ (\overline{A}_{ao}+\overline{A}_1) \\ \overline{C}_a \end{bmatrix} \geq 0 \tag{4.105}$$

If a solution exists, then $\exists Z \geq 0$ and can be rewritten by using full rank decomposition as follows;

$$Z = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U & V \end{bmatrix}^* \tag{4.106}$$

$$\left\{(\overline{A}_{ao}^T U + \overline{C}_a^T V) + \frac{\beta}{\gamma} U\right\} \left\{(\overline{A}_{ao}^T U + \overline{C}_a^T V) + \frac{\beta}{\gamma} U\right\}^* - c^2 U U^* \leq 0 \tag{4.107}$$

$$\left\{((\overline{A}_{ao}^T + \overline{A}_1^T) U + \overline{C}_a^T V) + \frac{\beta}{\gamma} U\right\} \left\{((\overline{A}_{ao}^T + \overline{A}_1^T) U + \overline{C}_a^T V) + \frac{\beta}{\gamma} U\right\}^* - c^2 U U^* \leq 0 \tag{4.108}$$

$$\vdots$$

$$\left\{((\overline{A}_{ao}^T + \overline{A}_k^T) U + \overline{C}_a^T V) + \frac{\beta}{\gamma} U\right\} \left\{((\overline{A}_{ao}^T + \overline{A}_k^T) U + \overline{C}_a^T V) + \frac{\beta}{\gamma} U\right\}^* - c^2 U U^* \leq 0 \tag{4.109}$$

Following definitions are required for further investigation;

$$F_i \triangleq \left\{((\overline{A}_{ao}^T + \overline{A}_i^T) U + \overline{C}_a^T V) + \frac{\beta}{\gamma} U\right\}, i = 1, 2, \ldots, k \tag{4.110}$$

$$G \triangleq cU \tag{4.111}$$

As it is stated in the previous section it is not possible to find a common W, such that $WW^* \leq I$, and

$$F_1 = GW \tag{4.112}$$

$$F_2 = GW \tag{4.113}$$

$$\vdots$$

$$F_k = GW \tag{4.114}$$

In order to find a common eigenvector for all matrix family, similar design steps can be followed. Note that, with the above definitions, Theorem 4.2 can be applied directly. If there exists a common ξ for all inequalities given below,

$$\begin{aligned} F_i\left(\sigma_1\xi_1\xi_1^* + \ldots + \sigma_{n+m}\xi_{n+m}\xi_{n+m}^*\right) F_i^* \\ -G(\sigma_1\xi_1\xi_1^* + \ldots + \sigma_{n+m}\xi_{n+m}\xi_{n+m}^*)G^* \leq 0 \end{aligned} \tag{4.115}$$

$$\begin{aligned} \sigma_1\left\{F_i\left(\xi_1\xi_1^*\right)F_i^* - G\left(\xi_1\xi_1^*\right)G^*\right\} + \ldots \\ +\sigma_{n+m}\left\{F_i\left(\xi_{n+m}\xi_{n+m}^*\right)F_i^* - G\left(\xi_{n+m}\xi_{n+m}^*\right)G^*\right\} \leq 0 \end{aligned} \tag{4.116}$$

where, $i = 1, 2, \ldots, k$. Then ξ is the candidate vector to satisfy the following statement for all $\overline{A}_j$, $j = 1, \ldots, k$

$$\begin{bmatrix} \lambda_i I - \left(\overline{A}_{ao}^T + \overline{A}_j^T\right) & -\overline{C}_a^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0 \tag{4.117}$$

where, $\lambda_i \in D$, $u \triangleq U\xi$, $v \triangleq V\xi$.

4.5 Case Studies

This section explores three case studies. The first study examines a nominal system with no uncertainties, while the second example considers an observable system with uncertainties. Finally, the third case study explores the reduced order system scenario.

4.5.1 Case-1: Design example for nominal system

Now let us use the proposed method for the nominal system. Let A and B be given as follows;

$$A = \begin{bmatrix} -3 & 3.1 \\ -3.1 & -3 \end{bmatrix},\ B = \begin{bmatrix} 0.5 \\ 0.6 \end{bmatrix} \tag{4.118}$$

and $\Theta \in H_2$ be given by

$$\Theta = \begin{bmatrix} -33.97 & -5+3.1j \\ -5-3.1j & -1 \end{bmatrix} \tag{4.119}$$

This means that the eigenvalue region is defined as a disc centered at $-5 \pm 3.1j$ with radius 0.8. By solving the following inequality,

$$\begin{bmatrix} I & 0 & A & B \end{bmatrix} (\Theta \otimes Z) \begin{bmatrix} I \\ 0 \\ A^* \\ B^* \end{bmatrix} \geq 0 \tag{4.120}$$

Z is found to be as follows;

$$Z = \begin{bmatrix} 0.1247 & 0.0064 - 0.206j & -0.4329 + 0.5056j \\ * & 0.3537 & -0.9578 - 0.7293j \\ * & * & 4.4512 \end{bmatrix} \tag{4.121}$$

Eigenvalues of Z are given below;

$$\lambda(Z) = \begin{bmatrix} 0 \\ 0.0488 \\ 4.9907 \end{bmatrix} \tag{4.122}$$

So, Z is clearly positive semi-definite. Let z be re-written as follows;

$$Z = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U^* & V^* \end{bmatrix} \tag{4.123}$$

where

$$U = \begin{bmatrix} -0.3201 & 0.1491 & -0.037 \\ -0.0314 - 0.5745j & -0.0248 + 0.1482j & -0.0005 + 0.00035j \end{bmatrix} \tag{4.124}$$

$$V = \begin{bmatrix} 1.3683 + 1.6047j & 0.0338 + 0.0535j & 0.0001 + 0.0004j \end{bmatrix} \tag{4.125}$$

Then,

$$F = (AU + BV) + \frac{\beta}{\gamma} U \tag{4.126}$$

$$G = cU \tag{4.127}$$

Solution for $FPF^* \leq GPG^*$ is given below;

$$P = \begin{bmatrix} 2.1873 & 0.0638 - 0.0127j & -0.0008 + 0.007j \\ * & 1.7518 & 0.0250 - 0.0783j \\ * & * & 0.8264 \end{bmatrix} \tag{4.128}$$

$$\lambda(P) = \begin{bmatrix} 0.8190 \\ 1.7496 \\ 2.1969 \end{bmatrix} \tag{4.129}$$

Clearly, P is positive definite. Therefore, the SVD of P can be written as follows;

$$P = U_p S_p U_p^* \tag{4.130}$$

where

$$S_p = \begin{bmatrix} 2.1969 & 0 & 0 \\ 0 & 1.7496 & 0 \\ 0 & 0 & 0.8190 \end{bmatrix} \tag{4.131}$$

$$U_p = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \end{bmatrix} \tag{4.132}$$

$$= \begin{bmatrix} -0.9894 & 0.1451 & -0.0093 \\ -0.1425-0.0283j & -0.9666-0.1917j & 0.0867+0.0174j \\ -0.0004+0.0036j & -0.01-0.0882j & -0.1103-0.989j \end{bmatrix} \tag{4.133}$$

In this case, the columns of U_p are the candidates for the solution. As suggested by the Theorem 4.2, at least one ξ has to satisfy the inequality of $||F\xi\xi^*F^*|| \leq ||G\xi\xi^*G^*||$. The solutions of $||F\xi_i\xi_i^*F^*|| \leq ||G\xi_i\xi_i^*G^*||$ for $i = 1,2,3$ are given below;

$$||F\xi_1\xi_1^*F^*|| - ||G\xi_1\xi_1^*G^*|| = -0.245 \tag{4.134}$$

$$||F\xi_2\xi_2^*F^*|| - ||G\xi_2\xi_2^*G^*|| = 0.1727 \tag{4.135}$$

$$||F\xi_3\xi_3^*F^*|| - ||G\xi_3\xi_3^*G^*|| = 0.0019 \tag{4.136}$$

Only ξ_1 satisfies the inequality, so the closed-loop eigenvectors can be found as follows;

$$u_1 \triangleq U\xi_1 = \begin{bmatrix} 0.2954-0.0042j \\ 0.0388+0.5480j \end{bmatrix} \tag{4.137}$$

$$v_1 \triangleq V\xi_1 = \begin{bmatrix} -1.3571-1.5962j \end{bmatrix} \tag{4.138}$$

$$u_2 = u_1^* \tag{4.139}$$

$$v_2 = v_1^* \tag{4.140}$$

Then,

$$R_n = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \tag{4.141}$$

$$Q_n = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \tag{4.142}$$

$$K = Q_n R_n^{-1} = \begin{bmatrix} -4.2066 & -2.0451 \end{bmatrix} \tag{4.143}$$

The assigned eigenvalues and D-Regions are shown in Figure 4.2.

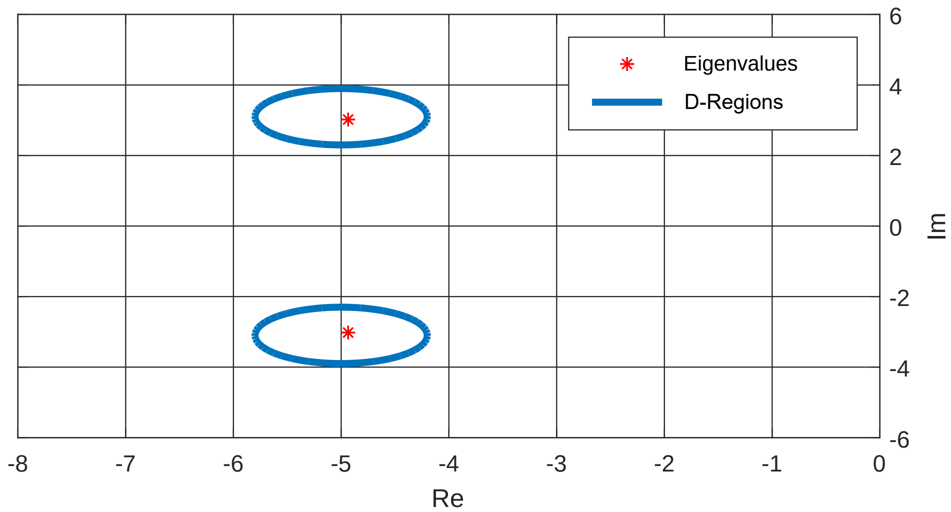

Figure 4.2 : The location of eigenvalues (Case 1: Nominal System).

4.5.2 Case 2: Design example for an uncertain system

In this case study, a DOB-based inner loop is designed in the presence of parametric uncertainty. Let A and B be given as follows;

$$A_0 = \begin{bmatrix} -3 & 3.1 & 0 \\ -3.1 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix} \tag{4.144}$$

$$A_1 = \begin{bmatrix} 0.9 & 0.9 & 0 \\ -0.9 & 0.9 & 0 \\ 0 & 0 & 0.4 \end{bmatrix} \tag{4.145}$$

$$A_2 = \begin{bmatrix} -0.9 & -0.9 & 0 \\ 0.9 & -0.9 & 0 \\ 0 & 0 & -0.4 \end{bmatrix} \tag{4.146}$$

$$B = \begin{bmatrix} 1 & 0.2 \\ 0.5 & 1 \\ 1 & 0.6 \end{bmatrix} \tag{4.147}$$

where, the matrix family is defined as; $A(\delta) = A_0 + \delta_1 A_1 + \delta_2 A_2$. The disturbance model is given below;

$$\dot{x_d} = A_d x_d \tag{4.148}$$

$$d = C_d x_d \tag{4.149}$$

where $x_d \in \mathbb{R}^{n_d}$

$$A_d = 0 \tag{4.150}$$

$$C_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.151}$$

Finally, the augmented state vector is $x_a = \begin{bmatrix} x^T & x_d^T \end{bmatrix}^T$, and augmented system matrices are,

$$A_a(\delta) = \begin{bmatrix} A(\delta) & \mathrm{B}\, C_d \\ 0 & A_d \end{bmatrix}, \; B_a = \begin{bmatrix} B \\ 0 \end{bmatrix}, \; C_a = \begin{bmatrix} C & 0 \end{bmatrix} \tag{4.152}$$

and $\Theta_1, \Theta_2 \in H_2$ be given by

$$\Theta_1 = \begin{bmatrix} -39.36 & -6+3j \\ -6-3j & -1 \end{bmatrix} \tag{4.153}$$

$$\Theta_2 = \begin{bmatrix} -144 & -12+j \\ -12-j & -1 \end{bmatrix} \tag{4.154}$$

which mean that eigenvalue region Θ_1 is defined as a disc centered at $-6 \pm 3j$ with radius 2.5, and Θ_2 is defined as a disc centered at $-12 \pm 1j$ with radius 1. By utilizing the duality principle, the following LMIs are to be solved;

$$\begin{bmatrix} I & 0 & (A_0 + A_1)^* & C_a^T \end{bmatrix} (\Theta_1 \otimes Z) \begin{bmatrix} I \\ 0 \\ (A_0 + A_1) \\ C_a \end{bmatrix} \geq 0 \tag{4.155}$$

$$\begin{bmatrix} I & 0 & (A_0 + A_2)^* & C_a^T \end{bmatrix} (\Theta_1 \otimes Z) \begin{bmatrix} I \\ 0 \\ (A_0 + A_2) \\ C_a \end{bmatrix} \geq 0 \tag{4.156}$$

$$\begin{bmatrix} I & 0 & (A_0)^* & C_a^T \end{bmatrix} (\Theta_1 \otimes Z) \begin{bmatrix} I \\ 0 \\ (A_0) \\ C_a \end{bmatrix} \geq 0 \tag{4.157}$$

$$\begin{bmatrix} I & 0 & (A_0)^* & C_a^T \end{bmatrix} (\Theta_2 \otimes X) \begin{bmatrix} I \\ 0 \\ (A_0) \\ C_a \end{bmatrix} \geq 0 \tag{4.158}$$

$$Z \geq 0 \tag{4.159}$$

$$X \geq 0 \tag{4.160}$$

Z and X are found to be as follows;

$$Z = 10^{-7} \begin{bmatrix} 0.0524 & z_{12} & z_{13} & z_{14} & z_{15} \\ * & 0.0135 & z_{23} & z_{24} & z_{25} \\ * & * & 0.0896 & z_{34} & z_{35} \\ * & * & * & 0.0033 & z_{45} \\ * & * & * & * & 0.4414 \end{bmatrix} \tag{4.161}$$

where,

$$z_{12} = 0.0058 + 0.0222j \quad (4.162)$$

$$z_{13} = 0.0189 - 0.0459j \quad (4.163)$$

$$z_{14} = -0.0082 + 0.0103j \quad (4.164)$$

$$z_{15} = -0.1167 - 0.0716j \quad (4.165)$$

$$z_{23} = -0.0062 - 0.0077j \quad (4.166)$$

$$z_{24} = 0.0021 + 0.0041j \quad (4.167)$$

$$z_{25} = -0.04993 + 0.0225j \quad (4.168)$$

$$z_{34} = -0.0172 - 0.0031j \quad (4.169)$$

$$z_{35} = -0.0198 - 0.1604j \quad (4.170)$$

$$z_{45} = 0.0085 + 0.0374j \quad (4.171)$$

$$X = \begin{bmatrix} 0.1101 & x_{12} & x_{13} & x_{14} & x_{15} \\ * & 0.0259 & x_{23} & x_{24} & x_{25} \\ * & * & 0.1728 & x_{34} & x_{35} \\ * & * & * & 0.0049 & x_{45} \\ * & * & * & * & 5.997 \end{bmatrix} \quad (4.172)$$

where,

$$x_{12} = 0.053 + 0.0047j \quad (4.173)$$

$$x_{13} = 0.1376 - 0.0071j \quad (4.174)$$

$$x_{14} = -0.0231 + 0.0023j \quad (4.175)$$

$$x_{15} = -0.8030 - 0.0929j \quad (4.176)$$

$$x_{23} = 0.0658 - 0.0092j \quad (4.177)$$

$$x_{24} = -0.0109 + 0.0021j \quad (4.178)$$

$$x_{25} = -0.3940 - 0.0113j \quad (4.179)$$

$$x_{34} = -0.0291 + 0.0014j \quad (4.180)$$

$$x_{35} = -0.9932 - 0.1661j \quad (4.181)$$

$$x_{45} = 0.1649 + 0.0359j \quad (4.182)$$

Eigenvalues of Z and X are given below;

$$\lambda(Z) = 10^{-7}\begin{bmatrix} 0 \\ 0 \\ 0.004 \\ 0.040 \\ 0.556 \end{bmatrix}, \ \lambda(X) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 6.308 \end{bmatrix} \tag{4.183}$$

Let Z be re-written as follows;

$$Z = \begin{bmatrix} U_z \\ V_z \end{bmatrix} \begin{bmatrix} U_z & V_z \end{bmatrix}^* \tag{4.184}$$

Then,

$$F_1 \triangleq (A_a U_z + BV_z) + \frac{\beta_1}{\gamma} U_z \tag{4.185}$$

$$F_2 \triangleq (A_a + A_1) U_z + BV_z) + \frac{\beta_1}{\gamma} U_z \tag{4.186}$$

$$F_3 \triangleq (A_a + A_2) U_z + BV_z) + \frac{\beta_1}{\gamma} U_z \tag{4.187}$$

$$G_z = c_1 U_z \tag{4.188}$$

where $c_1 = \sqrt{-\det(\Theta_1)} = 2.5$. Considering Theorem 4.2, there exists a solution for P, that is,

$$P = \begin{bmatrix} 0.0233 & p_{12} & p_{13} & p_{14} & p_{15} \\ * & 0.0366 & p_{23} & p_{24} & p_{25} \\ * & * & 0.0944 & p_{34} & p_{35} \\ * & * & * & 0.0021 & p_{45} \\ * & * & * & * & 0.0066 \end{bmatrix} \tag{4.189}$$

where

$$p_{12} = 0.001 + 0.0005j \tag{4.190}$$

$$p_{13} = -0.0001 + 0.0018j \tag{4.191}$$

$$p_{14} = -0.0032 - 0.0014j \tag{4.192}$$

$$p_{15} = 0.0013 + 0.0026j \tag{4.193}$$

$$p_{23} = 0.0241 - 0.0034j \tag{4.194}$$

$$p_{24} = 0.0014 - 0.0025j \tag{4.195}$$

$$p_{25} = -0.0028 + 0.0013j \tag{4.196}$$

$$p_{34} = 0.0058 + 0.0017j \tag{4.197}$$

$$p_{35} = -0.0026 - 0.0058j \tag{4.198}$$

$$p_{45} = -0.0008 - 0.0014j \tag{4.199}$$

Since all eigenvalues of P are positive, the SVD of P can be written as follows;

$$P = U_p S_p U_p^* \tag{4.200}$$

where

$$U_p = \begin{bmatrix} \xi_{p1} & \xi_{p2} & \xi_{p3} & \xi_{p4} & \xi_{p5} \end{bmatrix} \tag{4.201}$$

In this case, columns of U_p are candidates for eigenvectors. As suggested by Theorem 4.2, at least one ξ_p has to satisfy the inequality of $||F\xi_p{\xi_p}^* F^*|| \leq ||G\xi_p{\xi_p}^* G^*||$. Only

$$\xi_{p3} = \begin{bmatrix} -0.9190 \\ 0.297 + 0.1509j \\ -0.1142 - 0.0914j \\ 0.1084 - 0.0301j \\ -0.0643 + 0.0797j \end{bmatrix} \tag{4.202}$$

satisfies the inequalities $||F_i\xi_{p3}{\xi_{p3}}^* F_i^*|| \leq ||G\xi_{p3}{\xi_{p3}}^* G^*||$ for all i.

Let X be re-written as follows;

$$X = \begin{bmatrix} U_x \\ V_x \end{bmatrix} \begin{bmatrix} U_x^* & V_x^* \end{bmatrix} \tag{4.203}$$

Then,

$$F_4 \triangleq \left\{ (A_a U_x + BV_x) + \frac{\beta_2}{\gamma} U_x \right\} \tag{4.204}$$

$$G_x \triangleq c_2 U_x \tag{4.205}$$

where $c_2 = \sqrt{-\det(\Theta_2)} = 1$, Considering the Theorem 4.2, there exists a solution for R, that is,

$$R = \begin{bmatrix} 2.3412 & r_{12} & r_{13} & r_{14} & r_{15} \\ * & 1.9221 & r_{23} & r_{24} & r_{25} \\ * & * & 1.3904 & r_{34} & r_{35} \\ * & * & * & 0.9692 & r_{45} \\ * & * & * & * & 0.1390 \end{bmatrix} \tag{4.206}$$

where

$$
\begin{aligned}
r_{12} &= -0.1208+0.0405j && (4.207)\\
r_{13} &= -0.0650+0.0365j && (4.208)\\
r_{14} &= -0.0110-0.0230j && (4.209)\\
r_{15} &= -0.1639-0.1803j && (4.210)\\
r_{23} &= -0.0342+0.0449j && (4.211)\\
r_{24} &= 0.0284+0.0456j && (4.212)\\
r_{25} &= 0.3408+0.0352j && (4.213)\\
r_{34} &= -0.0231+0.0235j && (4.214)\\
r_{35} &= 0.1626+0.1218j && (4.215)\\
r_{45} &= -0.0383+0.0978j && (4.216)
\end{aligned}
$$

Since all eigenvalues of R are positive, SVD of R can be written as follows;

$$R = U_R S_R U_R^* \tag{4.217}$$

$$U_R = \begin{bmatrix} \xi_{r1} & \xi_{r2} & \xi_{r3} & \xi_{r4} & \xi_{r5} \end{bmatrix} \tag{4.218}$$

In this case columns of U_R are candidates for eigenvectors.

As suggested by Theorem 4.2, at least one ξ_p has to satisfy the inequality of $||F_4\xi_r{\xi_r}^*F_4^*|| \leq ||G\xi_r{\xi_r}^*G^*||$.

Since $||F_4\xi_{r1}{\xi_{r1}}^*F_4^*|| - ||G\xi_{r1}{\xi_{r1}}^*G^*|| = -0.243$, ξ_{r1} given below satisfies the inequality $||F_4\xi_{r1}{\xi_{r1}}^*F_4^*|| \leq ||G\xi_{r1}{\xi_{r1}}^*G^*||$.

$$\xi_{r1} = \begin{bmatrix} -0.9368 \\ 0.3064+0.0383j \\ 0.0781+0.0213j \\ 0.01553-0.0152j \\ 0.1192-0.0760j \end{bmatrix} \tag{4.219}$$

Hence, the closed-loop eigenvectors can be found as follows;

$$
\begin{aligned}
u_1 &\triangleq U_Z\xi_{p3} = 10^{-4}\begin{bmatrix} 0.6949+0.0562j \\ 0.1295-0.3020j \\ 0.2904+0.5669j \\ -0.1086-0.1352j \end{bmatrix} && (4.220)\\
v_1 &\triangleq V_Z\xi_{p3} = 10^{-4}\begin{bmatrix} -1.6397+1.0083j \end{bmatrix} && (4.221)\\
u_2 &= u_1^* && (4.222)\\
v_2 &= v_1^* && (4.223)\\
u_3 &= U_X\xi_{r1} = \begin{bmatrix} 0.2995-0.0012j \\ 0.1492-0.0124j \\ 0.3673+0.0149j \\ -0.0604-0.0054j \end{bmatrix} && (4.224)\\
v_3 &= V_X\xi_{r1} = \begin{bmatrix} -2.2840+0.2633j \end{bmatrix} && (4.225)\\
u_4 &= u_3^* && (4.226)\\
v_4 &= v_3^* && (4.227)\\
& && (4.228)
\end{aligned}
$$

then,

$$
\begin{aligned}
R_n &= \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} && (4.229)\\
Q_n &= \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} && (4.230)
\end{aligned}
$$

$$
L = Q_n R_n^{-1} = \begin{bmatrix} -18.9317 \\ 17.1332 \\ -21.1235 \\ -142.1487 \end{bmatrix} \tag{4.231}
$$

The eigenvalue-spread of the proposed design is given in Figure 4.3. Note that, all dominant eigenvalues are restricted in the predetermined disjoint regions. Only nominal eigenvalues corresponding to the eigenvectors u_3 and u_4 are constrained in the predetermined regions and uncertain eigenvalues are not constrained in those regions. The comparison of the step responses of system with DOB and without DOB is given in Figure A.1 in Appendix A2. As shown in Figure A.1, the open-loop system without DOB is more sensitive to both parameter changes and unknown disturbance input. On the other hand, the step response of the uncertain system with DOB is close to the nominal step response even under the unknown disturbance inputs. Note that,

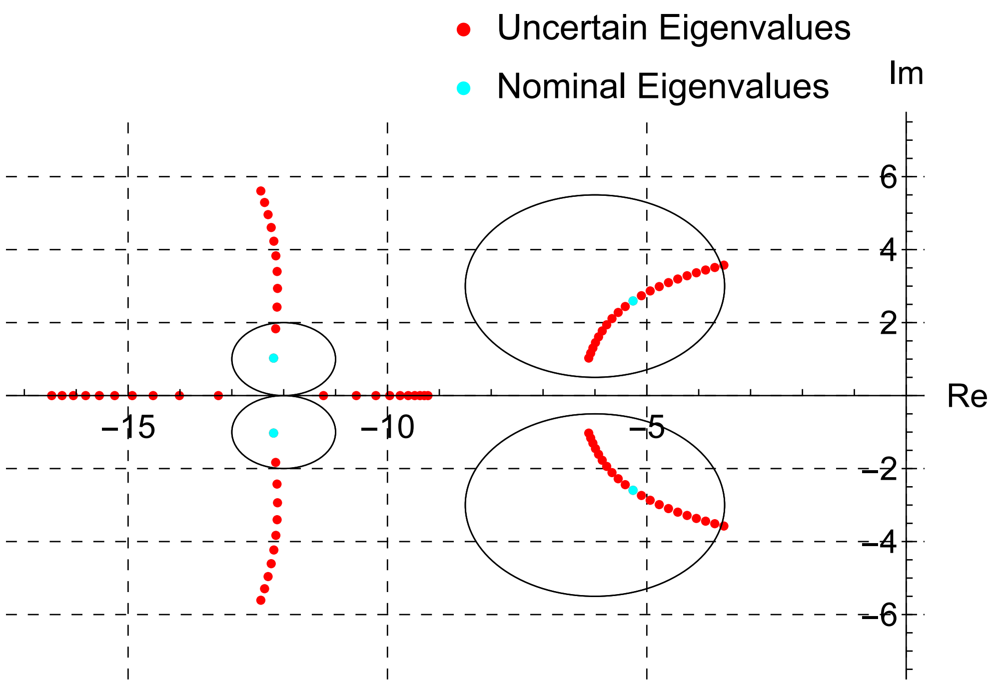

Figure 4.3 : The design by the proposed method (Case 2: DOB Design for uncertain system).

DOB estimates not only the external disturbances but also the effects of the parameter changes.

4.5.3 Case 3: Design example for reduced-order case

In this case study, a DOB-based inner loop is designed in the presence of parametric uncertainty. Let A and B be given as follows;

$$A_0 = \begin{bmatrix} -3 & 3.1 & 0 \\ -3.1 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix} \tag{4.232}$$

$$A_1 = \begin{bmatrix} 0.7 & 1.5 & 0 \\ -1.5 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \tag{4.233}$$

$$A_2 = \begin{bmatrix} -0.7 & -1.5 & 0 \\ 1.5 & 0 & 0 \\ 0 & 0 & -0.1 \end{bmatrix} \tag{4.234}$$

$$B = \begin{bmatrix} 1 & 0.2 \\ 0.5 & 1 \\ 1 & 0.6 \end{bmatrix} \tag{4.235}$$

$$C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \tag{4.236}$$

where, the matrix family is defined as; $A(\delta) = A_0 + \delta_1 A_1 + \delta_2 A_2$. The disturbance model is given below;

$$\dot{x_d} = A_d x_d \tag{4.237}$$

$$d = C_d x_d \tag{4.238}$$

where $x_d \in \mathbb{R}^{n_d}$

$$A_d = 0 \tag{4.239}$$

$$C_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.240}$$

Finally, the augmented state vector is $x_a = \begin{bmatrix} x^T \; x_d^T \end{bmatrix}^T$, and augmented system matrices are,

$$A_a(\delta) = \begin{bmatrix} A(\delta) & \mathrm{B}\, C_d \\ 0 & A_d \end{bmatrix},\; B_a = \begin{bmatrix} B \\ 0 \end{bmatrix},\; C_a = \begin{bmatrix} C & 0 \end{bmatrix} \tag{4.241}$$

and $\Theta_1, \Theta_2 \in H_2$ be given by

$$\Theta_1 = \begin{bmatrix} -21.61 & -4+3j \\ -4-3j & -1 \end{bmatrix} \tag{4.242}$$

$$\Theta_2 = \begin{bmatrix} -384 & -20 \\ -20 & -1 \end{bmatrix} \tag{4.243}$$

which mean that eigenvalue region Θ_1 is defined as a disc centered at $-4 \pm 3.1j$ with radius 2, and Θ_2 is defined as a disc centered at -20 with radius 4.

Note that, system (A_a, C_a) is not observable. Let us decompose the system into observable and unobservable subspaces.

Using similarity transformation, T, that is;

$$T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0.2289 & -0.2289 & 0 & 0.9462 \\ -0.6690 & 0.6690 & 0 & 0.3237 \\ 0.7071 & 0.7071 & 0 & \end{bmatrix} \tag{4.244}$$

$$\overline{A}_a = \begin{bmatrix} -6 & 0.9462 & 0.3237 & 0 \\ 0 & -0.2061 & 0.9559 & 1.0035 \\ 0 & 0.6024 & -2.7939 & -2.9331 \\ 0 & 0 & 3.2764 & -3 \end{bmatrix} \tag{4.245}$$

$$\overline{C}_a = \begin{bmatrix} 0 & 0 & 0 & 1.4142 \end{bmatrix} \tag{4.246}$$

Note that, $A_{a_{no}} = -6$, so only one mode of A_a is not observable. By utilizing duality principle, following LMIs are to be solved.

$$\begin{bmatrix} I & 0 & \left(\overline{A}_a^*\right)^* & \overline{C}_a^T \end{bmatrix} (\Theta_1 \otimes Z) \begin{bmatrix} I \\ 0 \\ \overline{A}_a \\ \overline{C}_a \end{bmatrix} \geq 0 \tag{4.247}$$

$$\begin{bmatrix} I & 0 & (\overline{A}_a + \overline{A}_1)^* & \overline{C}_a^T \end{bmatrix} (\Theta_1 \otimes Z) \begin{bmatrix} I \\ 0 \\ (\overline{A}_a + \overline{A}_1) \\ \overline{C}_a \end{bmatrix} \geq 0 \tag{4.248}$$

$$\begin{bmatrix} I & 0 & (\overline{A}_a + \overline{A}_2)^* & \overline{C}_a^T \end{bmatrix} (\Theta_1 \otimes Z) \begin{bmatrix} I \\ 0 \\ (\overline{A}_a + \overline{A}_2) \\ \overline{C}_a \end{bmatrix} \geq 0 \tag{4.249}$$

$$\begin{bmatrix} I & 0 & (\overline{A}_a)^* & \overline{C}_a^T \end{bmatrix} (\Theta_2 \otimes X) \begin{bmatrix} I \\ 0 \\ (\overline{A}_a) \\ \overline{C}_a \end{bmatrix} \geq 0 \tag{4.250}$$

$$Z \geq 0 \tag{4.251}$$

$$X \geq 0 \tag{4.252}$$

Z and X are found to be as follows;

$$X = \begin{bmatrix} 0 & -0.0001 & 0.0004 & -0.0037 \\ * & -0.0023 & -0.0107 & 0.1160 \\ * & * & 0.0513 & -0.5822 \\ * & * & * & 6.8923 \end{bmatrix} \tag{4.253}$$

$$Z = 10^{-3} \begin{bmatrix} -0.0106 & z_{12} & z_{13} & z_{14} \\ * & 0.4168 & z_{23} & z_{24} \\ * & * & 0.4965 & z_{34} \\ * & * & * & 0.5133 \end{bmatrix} \tag{4.254}$$

where

$$
\begin{aligned}
z_{12} &= -0.0485 - 0.0288j && (4.255)\\
z_{13} &= -0.0199 + 0.0568j && (4.256)\\
z_{14} &= 0.0428 - 0.0364j && (4.257)\\
z_{23} &= -0.1053 - 0.4536j && (4.258)\\
z_{24} &= -0.0857 + 0.4218j && (4.259)\\
z_{34} &= -0.4590 - 0.2234j && (4.260)
\end{aligned}
$$

Eigenvalues of Z and X are given below;

$$\lambda(Z) = 10^{-2}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.14 \end{bmatrix}, \ \lambda(X) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6.9435 \end{bmatrix} \tag{4.261}$$

Let Z be re-written as follows;

$$Z = \begin{bmatrix} U_z \\ V_z \end{bmatrix} \begin{bmatrix} U_z & V_z \end{bmatrix}^* \tag{4.262}$$

Then,

$$
\begin{aligned}
F_1 &\triangleq \left(A_{ao}^T U_z + C_a^T V_z\right) + \frac{\beta_1}{\gamma} U_z && (4.263)\\
F_2 &\triangleq \left((A_{ao}^T + A_1^T) U_z + C_a^T V_z\right) + \frac{\beta_1}{\gamma} U_z && (4.264)\\
F_3 &\triangleq \left((A_{ao}^T + A_2^T) U_z + C_a^T V_z\right) + \frac{\beta_1}{\gamma} U_z && (4.265)\\
G &\triangleq c_1 U && (4.266)
\end{aligned}
$$

where, $c_1 = \sqrt{-det(\Theta_1)} = 2$, considering the Theorem 4.2, there exists a solution for P s.t. $P > 0$. Therefore, SVD of P can be written as follows;

$$P = U_p S_p U_P^T \tag{4.267}$$

where

$$U_p = \begin{bmatrix} \xi_{p1} & \xi_{p2} & \xi_{p3} & \xi_{p4} \end{bmatrix} \tag{4.268}$$

$$\xi_{p1} = \begin{bmatrix} -0.0705 \\ -0.7355 + 0.5258j \\ -0.1463 - 0.3777j \\ 0.0966 - 0.0647j \end{bmatrix}, \ \xi_{p2} = \begin{bmatrix} 0.9811 \\ -0.0711 - 0.0117j \\ -0.0818 - 0.0016j \\ 0.0009 - 0.1596j \end{bmatrix} \tag{4.269}$$

$$\xi_{p3} = \begin{bmatrix} -0.1374 \\ 0.1226 - 0.2002j \\ -0.1062 - 0.3827j \\ -0.2886 - 0.8277j \end{bmatrix}, \ \xi_{p4} = \begin{bmatrix} 0.1163 \\ 0.2991 + 0.1810j \\ 0.4758 - 0.6672j \\ -0.2898 + 0.3297j \end{bmatrix} \tag{4.270}$$

In this case columns of U_p are candidate for eigenvectors. As suggested by Theorem 4.2, at least one ξ_p has to satisfy the inequality of $||F\xi_p\xi_p{}^*F^*|| \leq ||G\xi_p\xi_p{}^*G^*||$. Only ξ_{p2} satisfies the inequalities $||F_i\xi_{p2}\xi_{p2}{}^*F_i^*|| \leq ||G\xi_{p2}\xi_{p2}{}^*G^*||$ for all i, that is,

$$\begin{aligned} ||F_1\xi_{p2}\xi_{p2}{}^*F_1{}^*|| - ||G\xi_{p2}\xi_{p2}{}^*G^*|| &= -0.0034 \\ ||F_2\xi_{p2}\xi_{p2}{}^*F_2{}^*|| - ||G\xi_{p2}\xi_{p2}{}^*G^*|| &= -0.0005 \\ ||F_3\xi_{p2}\xi_{p2}{}^*F_3{}^*|| - ||G\xi_{p2}\xi_{p2}{}^*G^*|| &= -0.0021 \end{aligned} \tag{4.271}$$

Let X be re-written as follows;

$$X = \begin{bmatrix} U_x \\ V_x \end{bmatrix} \begin{bmatrix} U_x^* & V_x^* \end{bmatrix} \tag{4.272}$$

Then,

$$F_3 \triangleq \left(A_a^T U_x + C_a^T V_x\right) + \frac{\beta_2}{\gamma} U_x \tag{4.273}$$

$$G \triangleq c_2 U_x \tag{4.274}$$

where $c_2 = \sqrt{-det(\Theta_2)} = 4$. Considering the Theorem 4.2, there exist a solution for R such that $R > 0$. Therefore, SVD of R can be written as follows;

$$R = U_R S_R U_R^* \tag{4.275}$$

where,

$$U_R = \begin{bmatrix} \xi_{r1} & \xi_{r2} & \xi_{r3} & \xi_{r4} & \xi_{r5} \end{bmatrix} \tag{4.276}$$

In this case columns of U_R are candidate for eigenvectors. Then, at least one ξ_r has to satisfy the inequality of $||F_1\xi_r\xi_r{}^*F_1^*|| \leq ||G\xi_r\xi_r{}^*G^*||$.

ξ_{r1} given below satisfies the inequality $||F_1\xi_{r1}\xi_{r1}{}^*F_1^*|| \leq ||G\xi_{r1}\xi_{r1}{}^*G^*||$.

$$\xi_{r1} = \begin{bmatrix} -1 \\ -0.0023 \\ -0.007 \\ -0.0088 \end{bmatrix} \tag{4.277}$$

Finally, the closed-loop eigenvectors can be found as follows;

$$
\begin{aligned}
u_1 &\triangleq U_z\xi_{p2} = \begin{bmatrix} 0.0027+0.0007j \\ -0.0169+0.0107j \\ -0.008-0.0207j \end{bmatrix} && (4.278)\\
v_1 &\triangleq V_z\xi_{p2} = \begin{bmatrix} 0.0156+0.0151j \end{bmatrix} && (4.279)\\
u_2 &\triangleq u_1^* && (4.280)\\
v_2 &\triangleq v_1^* && (4.281)\\
u_3 &\triangleq U_X\xi_{r1} = \begin{bmatrix} 0.0014 \\ -0.0445 \\ 0.2217 \end{bmatrix} && (4.282)\\
v_3 &\triangleq V_X\xi_{r1} = \begin{bmatrix} -2.6252 \end{bmatrix} && (4.283)
\end{aligned}
$$

Then,

$$R_n = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \tag{4.284}$$

$$Q_n = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \tag{4.285}$$

$$K = \begin{bmatrix} 0 \\ Q_nR_n^{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ -145.2808 \\ -17.4654 \\ -14.4169 \end{bmatrix} \tag{4.286}$$

Note that, all dominant eigenvalues are restricted in the predetermined disjoint regions as it is seen in Figure 4.4.
Unobservable eigenvalues are not taken into account, but they are in the left half plane, so the design is valid.

The comparison of the step responses of system with DOB and without DOB is given in Figure A.2 in Appendix A3. As shown in Figure A.2, the open-loop system without DOB is more sensitive to both parameter changes and unknown disturbance input. On the other hand, the step response of the uncertain system with DOB is close to the nominal step response even under the unknown disturbance inputs.

4.6 Conclusion

This study proposes a new controller design method for uncertain systems via extended loose eigenstructure assignment. The study's main contribution is that it

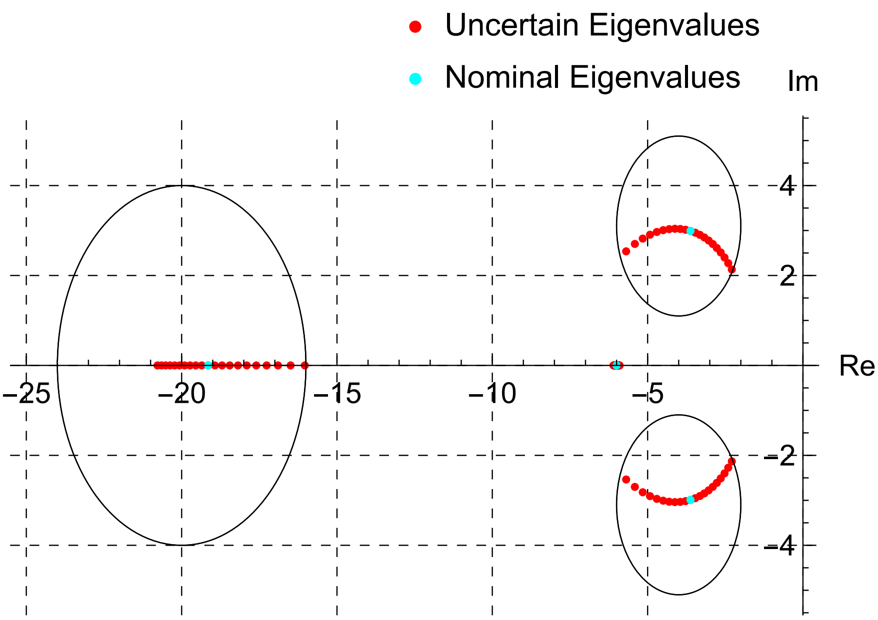

Figure 4.4 : Reduced-order design by proposed method (Case 3: Design example for reduced-order case).

allows handling robust root clustering problems for disjoint regions in the context of eigenstructure assignment. The concept of loose eigenstructure assignment has been enhanced to cover uncertain systems with acceptable conservatism. The method does not require any heuristic algorithms, and eigenvectors are selected among a finite number of vectors.

The concept has been applied to a disturbance observer design to restrict uncertain eigenvalues spread into predetermined disjoint regions. The method allows the flexibility of defining the different levels of measures for each nominal eigenvalue so that some eigenvalues can be left more relaxed, as it is carried out in the case study. Using the proposed method, an inner loop DOB-based system is designed to reject both external disturbances and the effect of internal uncertainties. Results showed that the behaviour of the uncertain system with DOB-based inner loop is close to the behaviour of the nominal system.

There are some cases where all states are not observable, or the designer may want to utilize low-order DOB, which is more desirable due to several reasons such as fast disturbance estimation and low complexity. To cover these kinds of design concerns, the reduced-order disturbance observer design will be studied in the future.

5. CONCLUSION

In this thesis, the disturbance observer-based control systems are analysed and various design methods are proposed for designing DOB-based control systems in the presence of parametric uncertainties. The research direction begins with the analysing DOB-based control systems in the frequency domain and spherical polynomials are used to represent uncertain dynamics. The value set concept is adopted to validate the results. In order to design a robust disturbance observer-based control system, two methodologies in which robustness is the major design criterion are proposed. Although the first method is useful to represent the disjoint uncertain regions and to constrain the uncertain eigenvalues within those regions, the method suffers from curse of dimensionality which leads to a computationally inefficient approach when the high dimensions are the case. On the other hand, the extended loose eigenstructure method is promising in that sense especially when dealing with reduced-order system dynamics.

The first research study within the thesis suggests a novel method for examining the DOB-based system by utilizing the spherical polynomial approach. The verification of the outcomes has been established through the adoption of the value set principle for spherical polynomials.

The main contributions of this study can be listed as follows: Firstly, the adoption of the spherical value set approach for uncertain polynomials has been made for the first time in disturbance observer-based control systems. Secondly, the robustness margin for a given DOB-based system has been systematically described in the context of spherical polynomial families, which has not been done before. Thirdly, the non-minimum phase case has been examined, and a detailed discussion has been made regarding the bandwidth constraints. Lastly, the effects of low-order DOB filter design have been investigated and discussed for potential impacts.

The analyses conducted in the first part of the study show the following results: If the nominal and uncertain plants have the same structure, and the uncertain parameters are present only on the denominator of the plant, then the robustness margin will increase as the DOB filter bandwidth increases (if the DOB bandwidth is higher than its minimum value). Additionally, the study demonstrates that when the nominal plant is first-order, irrespective of the order of the uncertain plant, the relation between robustness margin and DOB bandwidth is not straightforward. Therefore, it is not guaranteed that an increased DOB bandwidth will lead to an improved robustness margin in general. The same conclusion is valid for the affine linear case, where uncertain parameters are present both in the numerator and denominator of the plant. Lastly, the proposed method in this study enables the examination of the non-minimum phase case, in which the DOB-based systems may lose their superiorities, allowing for a more comprehensive analysis. The analytical derivation of the relation between the bandwidth and robustness of a disturbance observer has been carried out using a spherical polynomial representation.

The results obtained in the first part of the study have led to a new research direction. Specifically, the state space approaches are utilized to introduce novel design methods for the disturbance observer under parametric uncertainties. To this end, two new approaches are proposed. The first study presents a new approach to designing DOB for uncertain systems through the use of a novel guardian map. The main contribution of this study is that it allows for the constraint of uncertain eigenvalues into separate predefined D-regions. It is noteworthy that the predefined regions for nominal eigenvalues can differ from each other, thereby assigning different robustness criteria for each of the nominal eigenvalues. However, it is important to note that the method has a drawback, namely the curse of dimensionality, which requires further consideration in future work. As the dimension of the system increased, the computational effort also increased.

The final part of the study proposes a new controller design method for uncertain systems using a novel eigenstructure assignment within the LMI framework. The main contribution of this method is that it enables robust root clustering problems to be addressed for disjoint regions in the context of eigenstructure assignment. Notably, the concept of loose eigenstructure assignment has been improved to accommodate uncertain systems while maintaining acceptable conservatism. Furthermore, this method does not require the use of heuristic algorithms, and robust eigenvectors are selected from a finite number of alternatives.

The concept of extended loose eigenstructure assignment has been applied to the design of a disturbance observer to constrain the spread of uncertain eigenvalues into predetermined disjoint regions. The proposed method enables the flexibility of defining different levels of measures for each nominal eigenvalue, thereby allowing certain eigenvalues to be left more relaxed, as demonstrated in the case study. By utilizing this method, an inner loop DOB-based system has been designed to reject both external disturbances and the effect of internal uncertainties. The obtained results have demonstrated that the behavior of the uncertain system with a DOB-based inner loop is similar to the behavior of the nominal system.

It is acknowledged that there may be cases where not all states are observable or where the designer may prefer to use a low-order DOB due to advantages such as faster disturbance estimation and lower complexity. Therefore, to address these design concerns, another study is being carried out that focuses on the design of a reduced-order disturbance observer. The goal of this study is to provide a method for designing a DOB that can estimate disturbances effectively with fewer states while maintaining robustness against parametric uncertainties. Results from this study provide insights into the design of reduced-order DOB filters that can be applied to a wide range of practical engineering problems.

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APPENDICES

APPENDIX A : Lemma-3 of the Paper "On the Kalman-Yakubovich-Popov Lemma"
APPENDIX B : Step Responses With and Without DOB (Case 2: DOB Design For Uncertain System)
APPENDIX C : Step Responses With and Without DOB (Case 3: DOB Design For Reduced-Order Case)

APPENDIX A: Lemma-3 of the Paper "On the Kalman-Yakubovich-Popov Lemma"

Lemma [94] : Let F and G be complex matrices of the same size. Then;

1. $FF^* = GG^*$ if and only if there exists a matrix W such that $WW^* = I$ and $F = GW$.
2. $FF^* \leq GG^*$ if and only if there exists a matrix W such that $WW^* \leq I$ and $F = GW$.
3. $FG^* + GF^* = 0$ if and only if there exists a matrix W such that $WW^* = I$ and $F(I+W) = G(I-W)$.
4. $FG^* + GF^* \geq 0$ if and only if there exists a matrix W such that $WW^* \leq I$ and $F(I+W) = G(I-W)$.

Proof [94]: The statements (3) and (4) follow from (1) and (2) respectively, by replacement of F and G with $G-F$ and $F+G$. It remains to prove (1) and (2). Let the size of F and G be kxl. Consider first (1) for square matrices, i.e. the case $k = l$. Assuming that $FF^* = GG^*$, introduce the polar decompositions

$$F = H_F W_F \quad \textbf{(A.1)}$$
$$G = H_G W_G \quad \textbf{(A.2)}$$

where H_F and H_G are hermitian and positive semi-definite, while W_F and W_G are unitary. Then;

$$H_F = (FF^*)^{1/2} = (GG^*)^{1/2} = H_G \quad \textbf{(A.3)}$$

so the unitary matrix $W = W_G^* W_F$ satisfies $F = GW$.

The case $k < l$ follows immediately by extending F and G with zero rows to square matrices. If $k > l4$, then let F_1 be a submatrix of F with the same rank, but a minimal number of rows. Let G_1 be defined by the corresponding rows in G_1. Then $F_I F_1^* = G_1 G_1^*$ and existence of a unitary matrix W such that $F_1 = G_1 W$ follows as above. In fact, since all rows of F and G are linear combinations of the rows in F_1 and G_1, the desired equality $F = GW$ is proved as well.

To prove (2) from (1), note that $FF* \leq GG^*$, if and only if there exists an H such that $[F\ H][F\ H]^* = [G\ 0][G\ 0]^*$. By (1), this is equivalent to existence of H and a unitary matrix

$$\begin{bmatrix} W & V \\ V^* & X \end{bmatrix} \quad \textbf{(A.4)}$$

such that,

$$\begin{bmatrix} F & H \end{bmatrix} = \begin{bmatrix} G & 0 \end{bmatrix} \begin{bmatrix} W & V \\ V^* & X \end{bmatrix} \quad \textbf{(A.5)}$$

Such matrices exist if and only if $F = GW$ and $W^*W \leq I$. So (2) is proved.

□

APPENDIX B: Step Responses With and Without DOB (Case 2: DOB Design For Uncertain System)

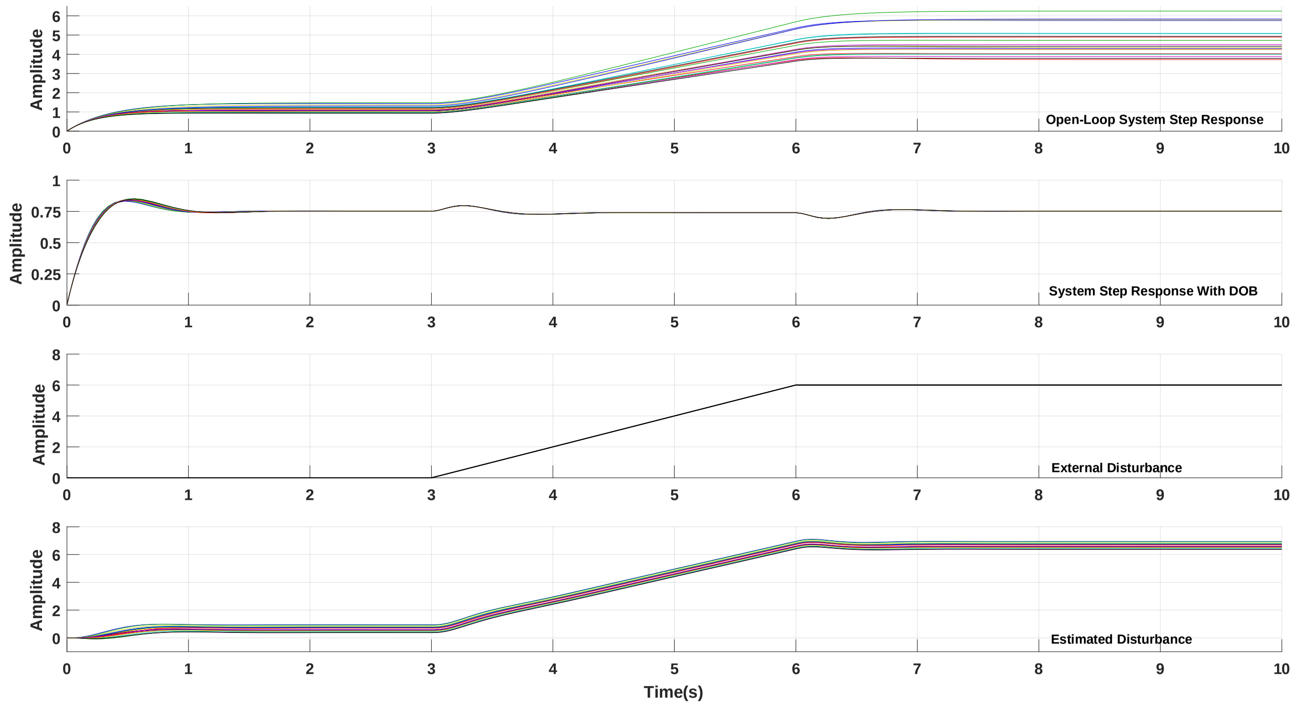

Figure A.1 : Step responses with and without DOB (Case 2: DOB design for uncertain system).

APPENDIX C: Step Responses With and Without DOB (Case 3: DOB Design For Reduced-Order Case)

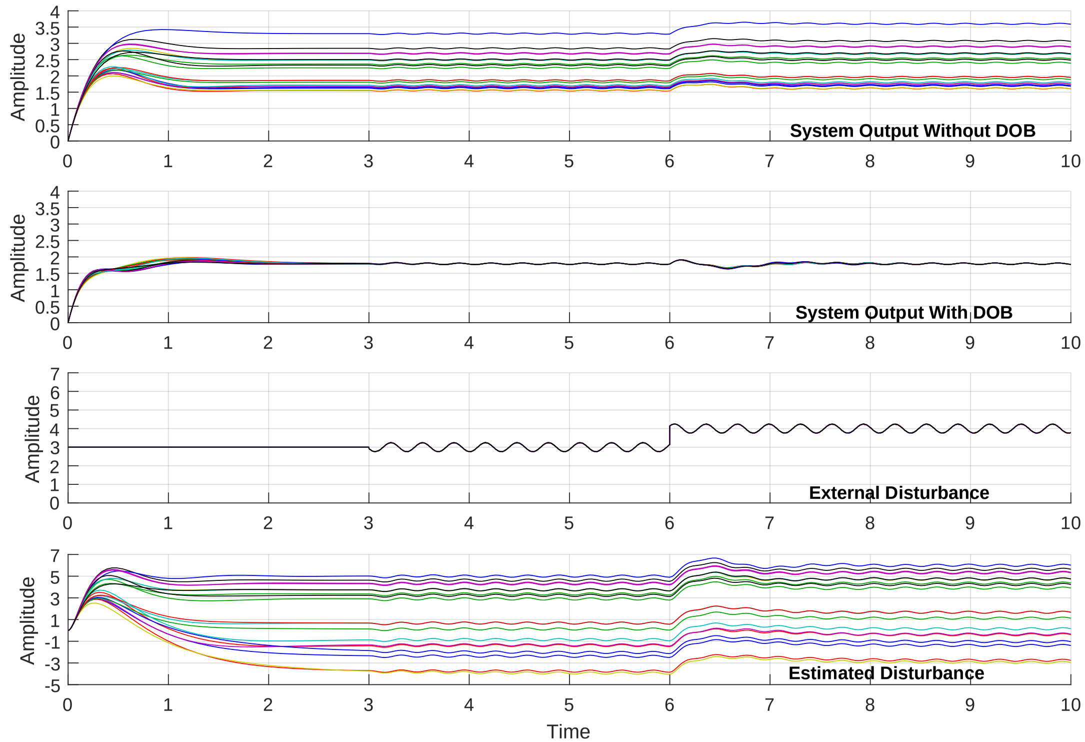

Figure A.2 : Step Responses with and without DOB (Case 3: DOB design for reduced-order case).

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